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DIFFUSION EQUATIONS IN DUALS OF NUCLEAR SPACES

by

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DIFFUSION EQUATIONS IN DUALS OF NUCLEAR SPACES

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Abstract: A stochastic Galerkin method is used to establish the existence of a solution to a martingale problem posed by an Itô type stochastic differential equation for processes taking values in the dual of a nuclear space. Uniqueness of the strong solution is also shown using the monotonicity condition. An application to the motion of random strings is discussed.

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1. Introduction. Let Φ be a nuclear Fréchet space and Φ' the strong topological dual space. We denote by $x[\varphi]$ the canonical pairing of elements $x \in \Phi'$, $\varphi \in \Phi$. Let $(\underline{F}_t)_{t<\infty}$ be a complete, right-continuous filtration on a complete probability space $(\Omega, \underline{F}, P)$. We will define below what is meant by an $\underline{F}_t$-Wiener martingale taking values in Φ', and give conditions on coefficient functions $A : \mathbb{R}_+ \times \Phi' \to \Phi'$ and $B : \mathbb{R}_+ \times \Phi' \to L(\Phi';\Phi')$ and on the probability distribution measure $\mu_0 = P \circ \xi^{-1}$ of a Φ'-valued random variable ξ under which we will prove the existence and uniqueness of solutions to stochastic differential equations (SDE's) of the form

$$dX_t = A_t(X_t)dt + B_t(X_t)dW_t, \qquad 0 \leq t < \infty; \qquad (1.1a)$$

with initial condition $P(X_0 = \xi) = 1$ or, in integrated form,

$$X_t = \xi + \int_0^t A_s(X_s)ds + \int_0^t B_s(X_s)dW_s \qquad 0 \leq t < \infty. \qquad (1.1b)$$

We begin by giving the definition of Φ and Φ' in Section 2 as well as preparatory material on Φ'-valued stochastic processes and the definitions of weak and strong solutions and of a solution to the martingale problem posed by (1.1). The conditions for existence of a weak solution and for uniqueness are given in Section 3. Section 4 is the pivotal section of the paper. It introduces the finite dimensional approximations to (1.1), produces a solution to the corresponding finite-dimensional martingale problem and obtains dimension-independent bounds for certain moments that are crucial to the Galerkin approximation. These results are used to prove the existence of a solution to the infinite dimensional martingale problem and to derive a weak solution (Sections 5 and 6). In Section 7, uniqueness is established by

proving the pathwise uniqueness property and using the, by now familiar, argument due to Yamada and Watanabe [4]. Our final result, Theorem 7.4 concerns the existence of a unique, Φ'-valued solution X_t for all $t \geq 0$. In the preceding results leading up to it, however, it is advantageous to restrict oneself to arbitrary finite intervals $[0,T]$, because the sample paths of the solution process will lie in Hilbert subspaces of Φ'. In general, the Hilbert spaces will depend on the value of T chosen and no Hilbert space will contain the sample paths for all time.

SDE's governing stochastic processes taking values in infinite dimensional linear spaces occur in such diverse fields as nonlinear filtering, infinite particle systems and population genetics. For many of these problems, the dual of a separable, Fréchet nuclear space provides a natural setting in which to study infinite dimensional martingales and SDE's.

Most of the applications known to us lead to linear equations, i.e. Ornstein-Uhlenbeck (Or Langevin) equations and their variants. (See [7] and [8] for references). An exception where Banach or Hilbert space-valued SDE's are concerned is the paper of Krylov and Rozovskii [8] which the present paper resembles in adapting the Galerkin approximation procedure to the stochastic context. An important difference is that here, the thrust of our efforts is first, to obtain a solution to the martingale problem. It calls for entirely different techniques which, moreover, do not involve the monotonicity condition. The latter is invoked only in proving uniqueness.

One of the main motivations for our interest in diffusion processes in duals of nuclear spaces is the possibility that they may provide a more realistic model to describe the behavior of the voltage potential of a spatially extended neuron. The celebrated, Hodgkin-Huxley deterministic theory

of such behavior takes into account many nonlinear features that are lacking in currently studied stochastic models (See [6] and references therein). A study just completed, of Φ'-SDE's driven by a discontinuous martingale in place of a Φ'-valued Wiener process suggests continuous approximations to diffusion equations similar to the ones discussed in this paper [5]. Weak convergence results of this kind would be of considerable use in the applications mentioned above. The investigation of these questions will be taken up in a later work. A more direct and immediate application, made in Section 8, is to the motion of random strings studied using different ideas, in [3].

2. Processes taking values in the dual of a nuclear Fréchet space. We shall present, in this section, preparatory material (including notation and terminology) on Fréchet nuclear spaces and on stochastic processes taking values in their dual spaces, leading up to the definition of stochastic differential equations governing such processes.

2.1 **Fréchet nuclear spaces.**

Throughout this paper Φ shall denote a fixed but arbitrary Fréchet nuclear space with strong dual Φ'. The topology of such a space can be given by an increasing family of semi-norms $\{|\cdot|_r\}_{-\infty<r<\infty}$ of the form $|\varphi|_r = [(\varphi,\varphi)_r]^{1/2}$ for a family of continuous symmetric scalar products $(\cdot,\cdot)_r$ on Φ such that the Hilbert-space completions H^r of Φ in the $|\cdot|_r$ of Φ in the $|\cdot|_r$ seminorms satisfy the following conditions:

(2.1.1) H^r and H^{-r} are canonically dual in the pairing whose restriction to $h^r \in \Phi \subseteq \Phi'$ is given by $\langle h^{-r}, h^r\rangle = h^{-r}[h^r]$. If the canonical mapping of H^r onto its dual H^{-r} is denoted j_r, then

$$\langle h^{-r}, h^r\rangle = h^{-r}[h^r] = (h^{-r}, j_r h^r)_{-r} = (j_{-r} h^{-r}, h^r)_r \ .$$

(2.1.2) $\Phi \subseteq H^r \subseteq H^s \subseteq \Phi'$ $\quad -\infty < s \leq r < \infty$, with

$\Phi = \bigcap_r H^r$ in the locally convex topology determined by $\{|\cdot|_r\}$,

$\Phi' = \bigcup_r H^{-r}$ in the inductive limit topology.

(2.1.3) For each r there exists a $p > r$ such that the injection mapping $i : H^p \to H^r$ is nuclear.

In the sequel we shall have to make sparing use of the completed tensor product of two nuclear Fréchet spaces $E \hat{\otimes} F$, which is also nuclear. The properties of the latter as well as additional details about nuclear spaces are to be found

in [12].

Fix any linearly-independent total set $\{\varphi_j\} \subset \Phi$ and, for each $s \geq 0$, let $\{h_j^s\}$ be the result of applying the Gram-Schmidt orthogonalization scheme to $\{\varphi_j\}$ in H^s. All finite linear combinations of the basis elements h_j^s lie in Φ, so for each d the d-dimensional space $H_d^s := sp\{h_1^s, \ldots, h_d^s\}$ is contained in Φ. For $s > 0$ let $\{h_j^{-s}\}_j \subset \Phi'$ be the associated dual basis for H^{-s} defined by the relation

$$h_j^{-s}[\varphi] := (h_j^s, \varphi)_s$$

for all $\varphi \in \Phi$, and let $H_d^{-s} := sp\{h_1^{-s}, \ldots, h_d^{-s}\}$. The spaces H^s and H^{-s} are canonically isomorphic under the mapping j_s. Denote by $\Pi_d^s u := \Sigma_{j\leq d}\, h_j^s h_j^{-s}[u]$ the orthogonal projection of an element $u \in H^s$ onto the d-dimensional subspace H_d^s, and by $\Pi_d^{-s} x := \Sigma_{j\leq d}\, x[h_j^s]h_j^{-s}$ the orthogonal projection of an element $x \in H^{-s}$ (or even $x \in \Phi'$) onto the d-dimensional subspace H_d^{-s}. For all $-\infty < s < \infty$ the space H_d^s is the image of $\mathbb{R}^d$ under the continuous injection $J_d^s : \mathbb{R}^d \to \Phi'$ given by

$$J_d^s[\xi] := \sum_{j=1}^{d} \xi_j h_j^s .$$

Note that Π_d^s and Π_d^{-s} are *dual* or *adjoint* in the sense that, for all $x \in \Phi'$ and $\varphi \in \Phi$,

$$x[\Pi_d^s \varphi] = \Sigma_{j\leq d}\, x[h_j^s]h_j^{-s}[\varphi] = (\Pi_d^{-s}x)[\varphi].$$

The space of continuous linear mappings from the Hilbert space H^r to the Hilbert space H^s will be denoted by $L(H^r;H^s)$, while the subspaces of nuclear (or trace-class) and Hilbert-Schmidt mappings will be denoted respectively by $L_1(H^r;H^s)$ and $L_2(H^r;H^s)$. The trace and trace norm of a mapping $A \in L_1(H^r;H^s)$ will be denoted by $tr(A)$ and $|A|_{r,s}$, respectively, while the Hilbert-Schmidt

norm of $A \in L_2(H^r;H^s)$ will be denoted by $\|A\|_{r,s}$.

In the sequel, probability measures will be studied on the spaces Φ' of continuous linear functionals on Φ and $C(\mathbb{R}_+;\Phi')$ of Φ'-valued (resp., $C(\mathbb{R}_+;H^{-p})$ of H^{-p}-valued) continuous functions on $[0,\infty)$, which will now be denoted by $C_{\Phi'}$ (resp., $C_{H^{-p}}$). Denote by $\underline{B}_{\Phi'}$ the Borel σ-algebra in Φ'; since Φ' is a countable inductive limit of Fréchet spaces, the Borel sets for Φ' endowed with the weak topology are the same as those for the strong topology. The Borel σ-algebra on $C_{\Phi'}$ and the induced σ-algebra on $C_{H^{-p}}$ will be denoted $\underline{B}_{C_{\Phi'}}$ and $\underline{B}_{C_{H^{-p}}}$, respectively.

2.2 Fréchet differentials in Φ'.

In order to study the generators of Φ'-valued diffusions we need first to give appropriate definitions of Fréchet differentials. For Hilbert and Banach spaces the concepts are well-known but, since we were unable to find a treatment of Fréchet differentials of functions on the dual of a Fréchet nuclear space in the literature, we give the definitions explicitly in this section.

Definition. Let $f : \Phi' \to \mathbb{R}$ be a continuous function. We call f *Fréchet differentiable at* $x \in \Phi'$ with (first) Fréchet differential $f'(x) \in L(\Phi';\mathbb{R})$ if for every $\epsilon > 0$ there exists a neighborhood U of 0 in Φ', open in the strong topology, such that for every index $m < \infty$ and element $h \in U \cap H^{-m}$,

$$|f(x+h) - f(x) - f'(x)[h]| < \epsilon |h|_{-m} . \tag{2.2.1}$$

If f has a Fréchet differential $f'(x)$ at x for every $x \in \Phi$, then for each m, the restriction of f to H^{-m} (denoted temporarily by f_m) has a Fréchet differential at each $x \in H^{-m}$ in the usual Hilbert-space sense. Furthermore, the derivative $f_m'(x)$ is the restriction to H^{-m} of $f'(x)$.

Again let f have a Fréchet differential $f'(x)$ at x for every $x \in \Phi'$, and suppose that the mapping $x \to f'(x)$ is continuous from $\Phi' \to L(\Phi';\mathbb{R}) \cong \Phi''$. We say that f is *second Fréchet differentiable at* $x \in \Phi'$ if the mapping $x \to f'(x)$ is itself Fréchet differentiable, or equivalently if there exists a map $f''(x) \in L(\Phi';\Phi') \cong B(\Phi',\Phi') \cong (\Phi' \otimes \Phi')'$ such that, for every $\epsilon > 0$, there exist neighborhoods U_1 and U_2 of 0 in Φ' such that for every pair m_1 and m_2 of indices, every $h_1 \in U_1 \cap H^{-m_1}$ and $h_2 \in U_2 \cap H^{-m_2}$,

$$|f(x+h_1+h_2) - f(x+h_1) - f(x+h_2) + f(x) - f''(x)[h_1,h_2]| \qquad (2.2.2)$$

$$< \epsilon |h_1|_{-m} |h_2|_{-m} .$$

Then $f''(x)$ is called the *second Fréchet differential of* f at x; we may regard it either as a bilinear form on Φ' or as a linear form on the tensor product $\Phi' \otimes \Phi'$. Again it can be shown that if $x \in H^{-m}$, the restriction of $f''(x)$ to $H^{-m} \times H^{-m}$ is the second Fréchet differential of $f_m(x)$.

The classes $\mathfrak{D}_b^2(\Phi')$ and $\mathfrak{D}_b^{1,2}(\Phi')$: $\mathfrak{D}_b^2(\Phi')$ denotes the vector space of all functions $f : \Phi' \to \mathbb{R}$ of the form

$$f(u) = \tilde{f}(u[\varphi]) \qquad (2.2.3)$$

for some bounded, twice continuously differentiable function $\tilde{f}$ on $\mathbb{R}$, ($\tilde{f} \in C_b^2(\mathbb{R})$) and $\varphi \in \Phi$.

The class $\mathfrak{D}_b^{1,2}(\Phi')$ consists of all functions $f : \mathbb{R}_+ \times \Phi' \to \mathbb{R}$ of the form $f_t(u) = \tilde{f}_t(u[\varphi])$ for some $\tilde{f} \in C_b^{1,2}(\mathbb{R}_+,\mathbb{R})$ and $\varphi \in \Phi$.

We will use the same symbol to denote the second class of functions even when t is restricted to a finite interval [0,T].

If f is given by (2.2.3) it is easily verified that f' and f" are given by the formulas

$$f'(u)[h] = \partial \tilde{f}(u[\varphi])h[\varphi] \qquad \text{for } h \in \Phi' \tag{2.2.4}$$

and

$$f''(u)[h_1,h_2] = \partial^2 \tilde{f}(u[\varphi])h_1[\varphi]h_2[\varphi] \qquad \text{for } h_1,h_2 \in \Phi'. \tag{2.2.5}$$

∂ and ∂^2 are the usual differentiation of the function $\tilde{f}$.

2.3 Φ'-valued processes and martingales:

Let $(\underline{F}_t)_{t<\infty}$ be a complete, right-continuous filtration on a complete probability space $(\Omega,\underline{F},P)$; all measurability conditions and martingale properties will be taken with respect to this fixed filtration. Integration with respect to P will usually be indicated with the expectation operator E, which will be denoted by E^P when the measure would otherwise be in doubt.

Definitions (i) A Φ'-valued random variable is an $\underline{F}_t/\underline{B}_{\Phi'}$-measurable mapping $\xi : \Omega \to \Phi'$.

(ii) A Φ'-valued, adapted process $M = (M_t)$ is called a martingale with respect to $(\underline{F}_t)$ if $M[\varphi] := (M_t[\varphi])$ is a real valued $(\underline{F}_t)$-martingale for each $\varphi \in \Phi$. The martingale M is called an L^2-martingale if, for each $\varphi \in \Phi$,

$$E\, M_t[\varphi]^2 < \infty \qquad \text{for } t \geq 0.$$

M is called a local (or local L^2) martingale if there is a sequence (τ_n) of $(\underline{F}_t)$-stopping times $\uparrow \infty$ a.s. such that, for each $\varphi \in \Phi$, $\{M_{t\wedge\tau_n}[\varphi]\}$ is a martingale (or L^2-martingale).

For a detailed discussion and properties of Φ'-martingales we refer the reader to [9] and [10]. We shall mention only those facts that will be directly useful for our purpose.

If M is a Φ' local martingale vanishing at the origin, there exists a unique $\Phi' \otimes \Phi'$ predictable process $\underline{A} = (\underline{A}_t)$ which is increasing (in the sense that for $\varphi \in \Phi$, the real valued process $\underline{A}_t[\varphi,\varphi]$ increases P-a.s.) and such that

$$Y_t[\varphi,\psi] := M_t[\varphi]M_t[\psi] - \underline{A}_t[\varphi,\psi]$$

is a local martingale satisfying $Y_0[\varphi,\psi] = 0$ P-a.s.

Definition: $\underline{A}$ is called the bracket function or the quadratic variation process of M and is often denoted by $\langle M,M_t\rangle$ or $\langle M\rangle_t$.

It is clear from the definition that

$$\langle M\rangle_t[\varphi,\psi] = \langle M_\bullet[\varphi],\ M_\bullet[\psi]\rangle_t$$

An important property of a Φ'-valued L^2-martingale with continuous paths is the following [9,10].

For each $T > 0$, there exists a positive number p (possibly depending on T) such that the sample paths lie in the Hilbert space H^{-p} for $0 \leq t \leq T$ and are continuous in the H^{-p} topology, i.e., $M_\bullet^T \in C^T_{H^{-p}}$ a.s. where $M_t^T = M_t$ for $0 \leq t \leq T$ and $C^T_{H^{-p}} := C([0,T];\ H^{-p})$.

The martingale with which we will almost exclusively, be concerned with in this paper is the one defined by a Φ'-valued Wiener process.

Definition: A Φ' *Wiener martingale* is a Φ' L^2 martingale W whose bracket function $\langle W\rangle$ is P-a.s. a non-random, linear (in t) function

$$\langle W\rangle_t[\varphi,\psi] \equiv t\ Q[\varphi,\psi]$$

for all $\varphi,\psi \in \Phi$. We call Q assumed continuous, the *covariance quadratic form* for W.

It follows from the remarks above that any Φ' Wiener martingale W may be taken to have paths which lie in the subspace H^{-q} for some $q < \infty$, and which are continuous in the H^{-q}-topology P-a.s.; the choice of q depends only on the quadratic form Q, and the sample paths will lie in H^{-s} for every $s \geq q$. The quadratic form Q has a unique continuous extension to a nuclear form on H^s,

and can be represented there in the form

$$Q[\varphi,\psi] = (\varphi,Q_s\psi)_s$$

$$= (Q_s^{1/2}\varphi,\ Q_s^{1/2}\psi)_s$$

for a unique non-negative trace-class operator Q_s on H^s. For later use, we choose and fix a specific value of s, say r. We denote the trace norm of the quadratic form Q (or, equivalently, of the operator Q_r^*) on H^r by

$$|Q|_{-r,-r} = \Sigma_j\ Q[h_j^r,h_j^r].$$

For any continuous quadratic form $Q[\cdot,\cdot]$ it is easy to construct a path-continuous Φ' process W_t which is a Wiener martingale (with bracket function $\langle W\rangle_t = tQ$) with respect to the filtration $(\underline{F}_t)_{t<\infty}$ generated by W; in Section 6 we face the more difficult task of constructing a Wiener martingale with respect to a given filtration $(\underline{F}_t)_{t<\infty}$.

2.4 Itô stochastic integrals in Φ'.

Because we are concerned primarily with diffusions in this paper, we shall briefly comment on the definition of one type of a stochastic integral with respect to a Φ'-valued Wiener process and describe some of its properties.

Let W be a Φ' Wiener martingale with continuous covariance quadratic form Q and hence bracket function

$$\langle W\rangle_t[\varphi,\psi] = t\ Q[\varphi,\psi].$$

The space of integrands, L_W^2 consists of those predictable functions $f_s : \mathbb{R}_+ \times \Omega \to L(\Phi';\Phi')$ for which

$$E \int_0^T Q[f_s^*\varphi,f_s^*\varphi]ds < \infty \qquad (2.4.1a)$$

for each $T > 0$ and each $\varphi \in \Phi$. Because it occurs frequently, we introduce the notation Q_A for the quadratic form given by

$$Q_A[\varphi,\psi] := Q[A^*\varphi, A^*\varphi]$$

for any continuous linear mapping $A \in L(\Phi';\Phi')$. In this notation we can rewrite (2.4.1a) as

$$E\,[\int_0^T Q_{f_s}[\varphi,\varphi]ds] < \infty. \tag{2.4.1b}$$

In the above, $A^* \in L(\Phi,\Phi)$ is the usual adjoint or dual of $A \in L(\Phi',\Phi')$.

The stochastic integral $I_t^f := \int_0^t f_s\, dW_s$, $(0 \le t \le T)$, is a Φ'-valued L^2-martingale with the following properties in addition to the usual linearity properties:

$$\langle I^t\rangle_t\,[\varphi,\psi] = \int_0^t Q_{f_s}[\varphi,\psi]ds. \tag{2.4.2}$$

There exists $m > 0$ (depending on f and T) such that

$$I_\bullet^f \in C^T_{H^{-m}} \quad \text{a.s.} \tag{2.4.3}$$

If $(h_i^m) \subset \Phi$ is any CONS in H^m,

$$I_t^f[\varphi] = \int_0^t f_s\, dWs_s[\varphi] = \sum_{i=1}^{\infty} \int_0^t (f_s^*\varphi, h_i^m)_m\, dW_s[h_i^m], \tag{2.4.3a}$$

the right hand side being an L^2-convergent series of ordinary Itô stochastic integrals. Furthermore,

$$\langle I^f\rangle_t[\varphi,\psi] = \sum_{i,j=1}^{\infty} \int_0^t (f_s^*\varphi,\ h_i^m)_m\,(f_s^*\psi, h_j^m)_m\, ds\ Q[h_i^m, h_j^m]. \tag{2.4.3b}$$

The Wiener processes $b_t^i := W_t[h_i^m]$ satisfy $\langle b^i, b^j\rangle_t = t\,Q[h_i^m, h_j^m]$, and they are independent if the set (h_i^m) diagonalizes Q.

2.5. Stochastic integrals for cylindrical Brownian motions.

Let H be a real separable Hilbert space with inner product $(\cdot,\cdot)_H$. A

cylindrical Brownian motion (or CBM) on H is a mapping $W^H : \mathbb{R}_+ \times H \to L^2(\Omega, \underline{F}, P)$ satisfying the following conditions:

(i) For all $h_1, h_2 \in H$ and $c_1, c_2 \in \mathbb{R}$, all $t \geq 0$,

$$W_t^H[c_1h_1 + c_2h_2] = c_1 W_t^H[h_1] + c_2 W_t^H[h_2] \quad \text{P-a.s.}$$

(ii) For each $h \in H$, $W_t^H[h]$ is a real-valued Wiener martingale with mean 0 and bracket function

$$\langle W^H\rangle[h_1], W^H[h_2]\rangle_t = t(h_1,h_2)_H .$$

It follows from (ii) that $E[(W_t^H[h])^2] = t(h,h)_H$ and hence that W_t^H cannot have sample-paths lying in H. For any complete orthonormal set $\{h_i\} \subset H$ we can produce independent real-valued standard Wiener martingales $b_t^i := W_t^H[h_i]$ and with them represent the CBM as the L^2 convergent series

$$W_t^H[h] = \sum_{i=1}^{\infty} (h,h_i)_H b_t^i . \qquad (2.5.1)$$

The relation between Φ' Wiener martingales and CBMs is given in the following propositon:

Proposition 2.5 Let W be a Φ' Wiener martingale with continuous covariance quadratic form Q, and let H^Q be the closure in Φ' of Φ in the norm $|\varphi|_Q :=$ $(Q[\varphi,\varphi])^{1/2}$. Then W has a unique $L^2(\Omega,\underline{F},P)$-continuous extension W^H to the Hilbert space $H = H^Q$, and W^H is a CBM on H^Q.

Conversely, any CBM W_t^H on a Hilbert space H satisfying $\Phi \subset H \subset \Phi'$ (with both inclusions continuous) determines a unique Φ' Wiener martingale, which may be given by the Φ' convergent series

$$W_t = \sum_{i=1}^{\infty} b_t^i h_i^* \qquad (2.5.2)$$

for any complete orthonormal sequence $\{h_i\} \subset \Phi \subset H$, where $b_t^i := W_i^H[h_i]$ and where $h_i^* \in \Phi'$ is the adjoint of h_i defined by

$$h_i^*[\varphi] = (h_i,\varphi)_H$$

for all $\varphi \in \Phi$.

Proof. Let H^* be the closure in Φ' of Φ in the inner product $(\varphi,\psi)_{H^*} := \sum_{i=1}^{\infty} h_i[\varphi]h_i[\psi]$, and let j be the canonical mapping of H onto its dual space H^*; now verify that (2.5.2) converges P-a.s. in the K topology for any Hilbert space $K \subset \Phi'$ such that $H \subset K$ with the inclusion mapping nuclear. ∎

Fix a real separable Hilbert space K and let $L^2_{H,K}(W^H)$ denote the linear space of $L_2(H;K) \cong H \otimes K$ - predictable processes

$$f : \mathbb{R}_+ \times \Omega \to L_2(H;K) \tag{2.5.3}$$

satisfying the condition

$$E\,[\int_0^T \|f_s\|^2_{H,K}\, ds] < \infty$$

for each $T \in \mathbb{R}_+$. For almost every $t \geq 0$, f_t is (almost surely) a Hilbert-Schmidt operator from H to K with an adjoint f_t^* Hilbert-Schmidt from K to H. Then for each $t > 0$, $k \in K$, and index i, the Itô stochastic integral

$$\int_0^t (f_s^*(\omega)k,h_i)_H db_s^i(\omega)$$

is well-defined and so is the $L^2(\Omega,\underline{F},P)$ convergent sum

$$I_t[k](\omega) := \sum_{i=1}^{\infty} \int_0^t (f_s^*(\omega)k,h_i)_H\, db_s^i(\omega).$$

An easy computation shows that $E[I_t[k]] = 0$ and

$$E[I_t[k]^2] = \sum_{i=1}^{\infty} \int_0^t E[(f_s^* k, h_i)_H^2]ds = \int_0^t E[|f_s^* k|_H^2]ds,$$

so for any complete orthonormal system $\{k_j\}$ in K,

$$E[\sum_{j=1}^{\infty} I_t[k_j]^2] = \int_0^t E[\sum_{j=1}^{\infty} |f_s^* k_j|_H^2]ds = E[\int_0^t \|f_s^*\|_{K,H}^2 \, ds] = E[\int_0^t \|f_s\|_{H,K}^2 \, ds] < \infty.$$

With probability one the real-valued series $\sum_{j=1}^{\infty} I_t[k_j]^2$ converges and hence so does the K-valued series

$$I_t := \sum_{j=1}^{\infty} I_t[k_j]k_j \ .$$

Definition: The *cylindrical stochastic integral of* f_s with respect to the CBM W_s^H is the sum

$$\int_0^t f_s(\omega) \, dW_s^H(\omega) := \sum_{j=1}^{\infty} I_t[k_j]k_j \ ,$$

or zero on the P-null set on which the series fails to converge.

The cylindrical stochastic integral of a process $\int_0^t f_s \, dW_s^H$ is a K-valued process (denoted I_t above), whose inner product with an element $k \in K$ we write (following [13]) as

$$\int_0^t (f_s^* k, \, dW_s^H)_H := (I_t, k)_K \ ;$$

thic can be calculated as the convergent series

$$\int_0^t (f_s^* k, \, dW_s^H)_H = \sum_{i=1}^{\infty} \int_0^t (f_s^* k, \, h_i)_H \, db_s^i$$

of real-valued Itô integrals.

2.6 Stochastic differential equations for Φ' processes: Solutions of martingale problems, weak and strong solutions.

Fix a Fréchet nuclear space Φ, a continuous quadratic for Q on the dual space Φ', a probability measure μ_0 on the Borel sets $\underline{B}(\Phi')$, and a pair of continuous functions

$$A : \mathbb{R}_+ \times \Phi' \to \Phi'$$

and

$$B : \mathbb{R}_+ \times \Phi' \to \Phi' \otimes \Phi,$$

and recall from Section 1 the stochastic integral equation

$$X_t = \xi + \int_0^t A_s(X_s)ds + \int_0^t B_s(X_s)dW_s, \qquad 0 \leq t < T. \tag{1.1b}$$

Definition. In the spirit of [4] we define a (weak) *solution* to equation (1.1b) on the interval [0,T] to be any Φ' process $X := (X_t)_{t\in\mathbb{R}_+}$ on any complete probability space $(\Omega,\underline{F},P)$ with a complete, right-continuous filtration $(\underline{F}_t)_{t<\infty}$ satisfying the conditions:

(2.6.1a) The Φ' random variable X_0 has probability distribution $\mu_0 = P \circ \xi^{-1}$;

(2.6.1b) There exists a Φ' Wiener martingale W for the given filtration $(\underline{F}_t)_{t<\infty}$ on $(\Omega,\underline{F},P)$;

(2.6.1c) X is adapted to $(\underline{F}_t)_{t<\infty}$, i.e. for every $t \in \mathbb{R}_+$ and $\varphi \in \Phi$ the random variable $X_t[\varphi]$ is $\underline{F}_t/\underline{B}(\mathbb{R})$ measurable;

(2.6.1d) For P-a.e. ω, X has strongly continuous paths $X_t(\omega) : \mathbb{R}_+ \to \Phi'$;

(2.6.1e) The predictable process $\alpha_t := A_t(X_t)$ satisfies $\alpha_t[\varphi] \in L^1([0,T])$ P-a.s. for each $\varphi \in \Phi$, i.e.

$$\int_0^T |\alpha_s|[\varphi]|ds < \infty;$$

(2.6.1f) For a sequence τ_n of stopping times converging P-a.s. to infinity the predictable process $\beta_t := B_{t\wedge\tau_n}(X_{t\wedge\tau_n})$ satisfies for each $\varphi \in \Phi$ and each $t \leq T$,

$$E[\int_0^{t\wedge\tau_n} Q_{\beta_s}[\varphi,\varphi]ds] < \infty;$$

(2.6.1g) With probability one, X_t and W_t satisfy (1.1b) for $0 \leq t \leq T$.

Note that although (1.1b) is only required to hold on the interval $[0,T]$, X is defined on the entire positive half-line $\mathbb{R}_+$ and so induces a probability measure $\mu = P \circ X^{-1}$ on the canonical space $C(\mathbb{R}_+;\Phi')$. We may define X_t initially only for $0 \leq t \leq T$ and then set $X_t = X_T$ for $t > T$ if convenient.

Definition: A *strong solution* is just like a weak solution, except that we specify the probability space $(\Omega,\underline{F},P)$, filtration $(\underline{F}_t)_{t<\infty}$, $\underline{F}_0$-measurable initial random variable ξ with probability distribution μ_0, and Wiener martingale W with covariance functional Q at the outset. To produce a *strong solution* we must construct the process X_t on the given space and for the given filtration.

Definition: A solution for $0 \leq t \leq T$ to the *martingale problem* posed by (1.1b) is a probability measure μ on the Borel sets of the canonical space $C_{\Phi'} := C(\mathbb{R}_+;\Phi')$ of Φ'-valued continuous functions on $[0,\infty)$ such that, for any real-valued function $f \in \mathfrak{D}_b^2(\Phi')$, the real-valued process

$$M_t^f := f(x_{t\wedge T}) - f(x_0) - \int_0^{t\wedge T} L_s\, f(x_s)ds \tag{2.6.2}$$

is a $(C_{\Phi'},(\underline{F}_t)_{t<\infty},\mu)$ martingale and $\mu \circ x_0^{-1} = \mu_0$, i.e. for all $f \in \mathfrak{D}_b^2(\Phi')$,

$$E^{\mu}[f(x_0)] = \int_{\Phi'} f(x)\, \mu_0(dx) \tag{2.6.3a}$$

and, for bounded $s \leq t \leq T$ and $\underline{F}_s$-measurable functions g,

$$0 = E^{\mu}[g(x)[M_t^f - M_s^f]] \tag{2.6.3b}$$

$$= E^{\mu}[g(x)[f(x_t) - f(x_s) - \int_s^t L_u\, f(x_u)du]].$$

Here L_s denotes the generator

$$L_s\, f(u) := f'(u)[A_s(u)] + \tfrac{1}{2} f''(u)[Q_{B_s(u)}] \tag{2.6.4}$$

$$= \tilde{f}'(u[\varphi])A_s(u)[\varphi] + \tfrac{1}{2} \tilde{f}''(u[\varphi])Q(B_s^*(u)\varphi,\ B_s^*(u)\varphi)$$

which is well-defined for $f \in \mathfrak{D}_b^2(\Phi')$. We describe this as the martingale problem "suggested by (1.1b)" since (by Itô's formula) the measure μ induced on $C_{\Phi'}$ by any weak solution X to (1.1b) does in fact satisfy (2.6.2), so that a solution to the martingale problem is just the marginal distribution measure μ for a weak solution X_t to (1.1b).

It is well-known (see, for example, [11]) that (2.6.2) is equivalent to the apparently stronger time-dependent form requiring that, for any real-valued function $f \in \mathfrak{D}_b^{1,2}(\Phi')$, the real-valued process

$$M_t^f := f_{t\wedge T}(x_{t\wedge T}) - f_0(x_0) - \int_0^{t\wedge T} (\partial/\partial s + L_s)f_{s\wedge T}(x_s)ds \tag{2.6.2'}$$

be a $(C_{\Phi'}, (\underline{F}_T)_{t<\infty}, \mu)$ martingale.

A statement equivalent to saying that μ is a solution to the martingale problem is that, for every $\varphi \in \Phi$,

$$M_t[\varphi] := x_t[\varphi] - x_0[\varphi] - \int_0^t A_s(x_s)[\varphi]ds$$

is a $(C_{\Phi'}, \underline{F}_t, \mu)$ local martingale.

3. Conditions for existence and uniqueness. First of all we shall impose a basic conditon on the nuclear space Φ. To introduce the condition we begin with the following observation: Let $(h_j^m) \subset \Phi$ be a CONS in H^m. The h_j^m can be obtained by applying the Gram-Schmidt orthonormalization procedure to a countable subset $\{\xi_j\}$ dense in Φ. For every j, we then have

$$\xi_j = \sum_{k=1}^{n_j} \alpha_{mk} h_k^m + \eta_j \tag{3.1}$$

where n_j (depending on m and j) $\leq$ j and $|\eta_j|_m = 0$.

Our basic assumption is the following:

(A) For each m and p, $(p \geq m)$, in the relation (3.1)

$$|\eta_j|_p = 0 . \tag{3.2}$$

Note that the relation (3.1) always holds but the possibility of satisfying (3.2) is a restriction on the type of nuclear spaces considered here. Condition A is of a technical nature. However, it is easy to see that it is satisfied if there exists a sequence $(\varphi_j) \subset \Phi$ which is a common orthogonal system in H^m for all $m \geq 1$. The Schwarz space $\mathcal{S}(\mathbb{R}^d)$ belongs to this class as well as the space Φ introduced in Section 8.

Condition (A) will be in force throughout the paper and will not be repeated in the statement of the results.

The following set of conditions will be needed to prove the existence of a solution to the martingale problem (and of a weak solution):

We assume given (i) a probability measure μ_0 on the Borel σ-field $\underline{B}_{\Phi'}$, (ii) a continuous quadratic form Q on $\Phi \times \Phi$ and (iii) coefficients A and B which, in addition to the measurability assumptions stated in the previous section

satisfy the following conditions:

For each $T > 0$ and sufficiently large $m \geq r$ (fixed above), there exists a number $\theta > 0$ and an index $p \geq m$ such that for all $s, t \leq T$,

(IC) Initial Condition:

$$c_0 := \int_{\Phi'} (1 + |u|^2_{-m}) [\log(3 + |u|^2_{-m})]^2 \mu_0(du) < \infty;$$

(CC) Coercivity Condition: for each $u \in j_m\Phi$,

$$2 A_t(u)[j_{-m}u] + |Q_{B_t(u)}|_{-m,-m} \leq \theta(1 + |u|^2_{-m});$$

(LG) Linear growth condition: if $u \in H^{-m}$, then $A_t(u) \in H^{-p}$ and

$$|A_t(u)|^2_{-p} \leq \theta(1 + |u|^2_{-m});$$

$$|Q_{B_t(u)}|_{-m,-m} \leq \theta(1 + |u|^2_{-m}) .$$

(JC) Joint continuity condition:

$$A : \mathbb{R}_+ \times \Phi' \to \Phi' \quad \text{and} \quad B : \mathbb{R}_+ \times \Phi' \to L(\Phi',\Phi')$$

is each jointly continuous.

Furthermore,

(i) $B_s(u)(v) \in H^{-m}$ if $u,v \in H^{-m}$ and

(ii) $Q(B^*_s(u)\varphi, B^*_s(u)\varphi)$ is continuous in u on Φ' for each $\varphi \in \Phi$.

In addition to the above, the following condition will be needed in the proof of uniqueness:

(MC) Monotonicity Condition: For all $u, v \in H^{-m}$ ($\subset H^{-p}$),

$$(A_t(u) - A_t(v), u-v)_{-p} + |Q_{B_t(u)-B_t(v)}|_{-p,-p} \leq \theta \, |u-v|^2_{-p} .$$

In the initial condition (IC) we have had to assume the finiteness of a moment of $|x_0|_{-m}$ slightly higher than the second. It is crucially used in solving the martingale problem for the infinite dimensional stochastic differential equation as well as the martingale problem for the finite dimensional approximation. The reason for a moment higher than the second is because we are not dealing with bounded coefficients and Lipschitz conditions but with linear growth and coercivity conditions.

4. Martingale Problems in Finite Dimensions. In this section we address the problem of the existence and uniqueness of a measure ν^d on the canonical space $C_{\mathbb{R}^d} = C(\mathbb{R}_+ : \mathbb{R}^d)$ of continuous paths in $\mathbb{R}^d$, equipped with the Borel sets $\underline{B}_{C_{\mathbb{R}^d}}$ for the compact open topology and the canonical filtration $(\underline{B}^t_{C_{\mathbb{R}^d}})$, solving the *martingale problem* [11] for the generator

$$L_s := \Sigma_i [a_s(\xi)]_i \partial_i + ½\, \Sigma_{i,j}\, [b^d_s(\xi) b^{d^*}_s(\xi)]_{ij}\, \partial_{ij} \qquad (4.1)$$

satisfying conditions outlined below. We apply these results in Section 5 to the problem of existence and uniqueness of a solution to an infinite-dimensional martingale problem (using a Galerkin method similar to that of [8]), but even in finite dimensions the bounds (4.3) of Theorem 4.1 seem to be new and may be of independent interest.

The martingale problem introduced above is closely related to the problem of the existence and uniqueness of solutions to the stochastic integral equation

$$X_t = X_0 + \int_0^t a_s(X_s) ds + \int_0^t b_s(X_s) dW_s$$

for an initial random variable X_0 with probability distribution ν^d_0 and a standard d-dimensional Wiener process W_t with covariance $E\, W_s W^*_t = (s \wedge t) I_d$. We return to this connection in the infinite-dimensional setting in Section 6, after proving a useful lemma whose proof is modeled after that of lemma 1.4.5 of [11].

Lemma 4.1. Let M be a continuous $\mathbb{R}^d$-valued L^2 martingale satisfying, for some $\beta < \infty$ and all $0 \leq s < t \leq T$, the inequality

$$\mathrm{tr}(\langle M \rangle_t - \langle M \rangle_s) \leq \beta(t-s).$$

Then for each ϵ there is a δ depending on β, ϵ, and T (and not d) such that

$$P[\sup_{\substack{0<s<\tau\leq T \\ t-s<\delta}} |M_t - M_s| > \epsilon] \leq \epsilon .$$

Proof. Set $\sigma_0 := 0$ and, for $n \geq 1$, $\sigma_n := \inf\{s > \sigma_{n-1} : |M_s - M_{\sigma_{n-1}}| > \epsilon/4\}$; also put $N := \inf\{n : \sigma_n > T\}$ and $\alpha := \inf\{\sigma_n - \sigma_{n-1} : 0 \leq n \leq N\}$. Note that $\{\sup_{\substack{0<s<\tau\leq T \\ t-s<\delta}} |M_t - M_s| > \epsilon\} \subset \{\alpha < \delta\}$, and so for any k,

$$P[\sup_{\substack{0<s<\tau\leq T \\ t-s<\delta}} |M_t - M_s| > \epsilon] \leq P[\alpha \leq \delta]$$

$$\leq P[N > k] + P[\sigma_n-\sigma_{n-1} < \delta \text{ for some } n \leq k] .$$

We now show that k may be chosen large enough to insure $P[N > k] < \epsilon/2$ and then δ small enough to force $P[\{\sigma_n-\sigma_{n-1} < \delta\}] < \epsilon/2k$ for all n. By Doob's inequality, for each $n \geq 1$, $t > 0$, and stopping time σ,

$$E[\sup_{0\leq s\leq T} |M_{s+\sigma} - M_\sigma|^2 \,|\underline{F}_\sigma] \leq 4\, E[|M_{t+\sigma} - M_\sigma|^2 \,|F_\sigma]$$

$$= 4\, E[tr(\langle M\rangle_{t+\sigma} - \langle M\rangle_s)|\underline{F}_s]$$

$$\leq 4\, \beta t .$$

It follows that

$$P[\sigma_n - \sigma_{n-1} < t|\underline{F}_{\sigma_{n-1}}] = P[\sup_{0\leq s\leq t} |M_{s+\sigma_{n-1}} - M_{\sigma_{n-1}}|^2 > (\epsilon/4)^2|\underline{F}_{\sigma_{n-1}}]$$

$$\leq 4\, \beta t/(\epsilon/4)^2$$

$$= 64\, \beta t/\epsilon^2$$

and that

$$E[e^{-(\sigma_n-\sigma_{n-1})}|\underline{F}_{\sigma_{n-1}}] = \int_0^\infty e^{-t}\, P[\sigma_n - \sigma_{n-1} < t|\underline{F}_{\sigma_{n-1}}]dt$$

$$\leq \int_0^\infty e^{-t} \min(1, 64\beta t/\varepsilon^2)dt \ .$$

If we denote the right hand by λ, then $\lambda < 1$ and

$$E[e^{-\sigma_k}] = E[e^{-\sigma_{k-1}} E[e^{-(\sigma_k-\sigma_{k-1})}|\underline{F}_{\sigma_{k-1}}] \leq \lambda\, E[e^{-\sigma_{k-1}}] \leq \lambda^k$$

(by induction) for each $k \geq 1$. By Chebyshev's inequality,

$$P[N > k] = P[\sigma_k \leq T] = P[e^{-\sigma_k} \geq e^{-T}] \leq e^T\lambda^k.$$

Pick k large enough to insure $e^T\lambda^k < \varepsilon/2$, and δ small enough that $k(64\ \beta\ \delta/\varepsilon^2) < \varepsilon/2$; then

$$P[\sigma_n-\sigma_{n-1} < \delta \quad \text{for some} \quad n \leq k] \leq k \max_{n\leq k} P[\sigma_n-\sigma_{n-1} < \delta]$$

$$\leq k(64\ \beta\ \delta/\varepsilon^2)$$

$$< \varepsilon/2\ ,$$

proving the lemma.

Definition 4.1.: A probability measure ν_0^d on $\mathbb{R}^d$ and continuous functions $a : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ satisfy the finite-dimensional initial condition (IC_d), coercivity condition (CC_d), and linear growth condition (LG_d) if for each $T < \infty$ there exist positive numbers c_0 and θ, not depending on d, such that, for all $t \leq T$ and $\xi\ \eta \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d}(1 + |\xi|^2)[\log(3 + |\xi|^2)]^2\ \nu_0^d(d\xi) \leq c_0 < \infty \qquad (IC_d)$$

$$2(\xi, a_t(\xi)) + |b_t^d(\xi)b_t^{d*}(\xi)| \leq \theta\ (1 + |\xi|^2) \qquad (CC_d)$$

$$|a_t^d(\xi)|^2 \le c_d(1 + |\xi|^2) \ , \qquad (LG_d)(a)$$

$$|b_t^d(\xi)b_t^{d*}(\xi)| \le \theta(1 + |\xi|^2) \ . \qquad (LG_d)(b)$$

Theorem 4.1. Let υ_0^d, a^d, and b^d satisfy conditions (IC_d), (LG_d), and (CC_d), and fix $T < \infty$. Then there exists a measure υ^d on $(C_{\mathbb{R}^d}, \underline{B}_{C_{\mathbb{R}^d}})$ satisfying the initial condition $\upsilon^d \circ x_0^{-1} = \upsilon_0^d$ and solving the martingale problem for L_s, i.e. satisfying the condition that for each $f \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ and $x \in C_{\mathbb{R}^d}$, the real-valued process

$$M_t^f(x) := f_{t\wedge T}(x_{t\wedge T}) - f_0(x_0) - \int_0^{t\wedge T} (\partial/\partial s + L_s)f_s(x_s)ds \qquad (4.2)$$

should be a local L^2 martingale on $(C_{\mathbb{R}^d}, \underline{B}_{C_{\mathbb{R}^d}}, \underline{B}^t_{C_{\mathbb{R}^d}}, \upsilon^d)$. Furthermore, for each $t \le T$ any such measure υ^d satisfies the following inequalities:

$$\upsilon^d[\sup_{0\le s\le t} |x_s| > R] \le \frac{2\, c_0\, e^{2\, c_3(\theta)T}}{(1+R^2)[\log(3+R^2)]^2} \qquad (4.3)$$

for every $R > 0$;

$$E^{\upsilon^d}[\sup_{0\le s\le t} f_1(x_s)] \le c_4(\theta) < \infty \qquad (4.4)$$

where $f_1(a) = (1 + a^2)\log\log(3+a^2)$,

$$c_4(\theta) := c_4(\theta;T) = 2c_0\, e^{2\, c_3(\theta)T} \int_0^\infty \frac{f_1'(a)}{(1+a^2)[\log(3+a^2)]^2}\, da$$

and

$$c_3(\theta) = 175\ \theta\ .$$

Proof of Theorem 4.1. For each $n \in \mathbb{N}$ define a function $c_n : \mathbb{R}^d \to \mathbb{R}^d$ and a

stopping time τ_n^d by

$$c_n(x) := \begin{cases} x \text{ if } |x| \le n , \\ nx/|x| \text{ if } |x| > n , \end{cases} \quad \tau_n^d(x) = \inf\{t \ge 0 : |x_t| \ge n \text{ or } t \ge T\}. \tag{4.5}$$

By Theorem 6.1.6 of [11] there exists a measure $\upsilon^{d,n}$ satisfying (4.2) with a^d and b^d replaced by the bounded continuous functions $a^{d,n}$ and $b^{d,n}$ given by

$$a_t^{d,n}(\xi_t) := a_t(c_n(\xi_t), \quad b_t^{d,n}(\xi_t) := b_t(c_n(\xi_t)) . \tag{4.6}$$

We will show that the sequence $\{\upsilon^{d,n}\}_{n\in\mathbb{N}}$ is tight and produce a limiting measure υ^d which satisfies (4.2), (4.3) and (4.4). Then, for $f(\xi) = (1 + |\xi|^2)[\log(3 + |\xi|^2)]^2$ ($\log(3 + |\xi|^2)$ is chosen instead of $\log(1 + |\xi|^2)$ since it is greater than equal to 1 for $|\xi|^2 \ge 0$), $M_{t\wedge\tau_n^d}$ given by (4.2) [with t replaced by $t \wedge \tau_n^d$] is a continuous $L^2 - \upsilon^{d,n}$ martingale. Using (CC_d) and (LG_d b), it is easy to verify that

$$|L_s f(\xi)| \le 15\theta\, f(\xi). \tag{4.7}$$

Since $a_t^{d,n}(x_t)$ and $b_t^{d,n}(x_t)$ coincide with $a_t^d(x_t)$ and $b_t^d(x_t)$ upto time τ_n^d, we have

$$M^t_{t\wedge\tau_n^d}(x) = f(x_{t\wedge\tau_n^d}) - f(x_0) - \int_0^t L_s\, f(x_{s\wedge\tau_n^d})ds \tag{4.8}$$

From (4.8) and (IC_d),

$$E^{\upsilon^{d,n}}[f(x_{t\wedge\tau_n^d})] \le c_0 + 15\theta \int_0^t E^{\upsilon^{d,n}}[f(x_{s\wedge\tau_n^d})]ds$$

and by Gronwall's inequality,

$$E^{\nu^{d,n}}[f(x_{t\wedge\tau_n^d})] \le c_0\, e^{15\theta t} \tag{4.9}$$

the bound being independent of n and d. Hence it is easily seen that $x_{t\wedge\tau_n^d}$ is an Itô process. Applying Itô lemma we have

$$\langle M^f\rangle_{t\wedge\tau_n^d} = \int_0^{t\wedge\tau_n^d} (\nabla f(x_s),\, b^{d,n}(x_s) b^{d,n}(x_s)^* \nabla f(x_s))ds$$

$$\le \int_0^{t\wedge\tau_n^d} |\nabla f(x_s)|^2\, |b_s^d(x_s)|^2\, ds\ .$$

From (CC_d) and the inequality

$$|\nabla f(\xi)|^2 \le 20\, f(\xi)\, [\log(3 + |\xi|^2)]^2$$

it follows that

$$\langle M^f\rangle_{t\wedge\tau_n^d} = (20)\theta \int_0^{t\wedge\tau_n^d} f(x_s)\, [\log(3 + |x_s|^2)]^2\, (1 + |x_s|^2)ds$$

$$\le (20)\theta \int_0^{t\wedge\tau_n^d} f(x_s)^2\, ds\ .$$

$$\le (20)\theta\, \{ \sup_{0\le s\le t\wedge\tau_n^d} f(x_s)\} \int_0^{t\wedge\tau_\nu^d} f(x_s)ds\ .$$

From Burkholder's inequality ([2], B, VII.92),

$$E^{\nu^{d,n}}[\sup_{0\le s\le t} |M^f_{s\wedge\tau_n^d}|] \le 4\, E^{\nu^{d,n}}[(\langle M^f\rangle_{t\wedge\tau_n^d})^{1/2}]$$

$$\leq \sqrt{320\theta}\, E^{\nu^{d,n}}\Big[\{\sup_{0\leq s\leq t} f(x_{s\wedge\tau_n^d})\}^{1/2} \{\int_0^{t\wedge\tau_n^d} f(x_s)ds\}^{1/2}\Big]$$

$$\leq \sqrt{320\theta}\, E^{\nu^{d,n}}\Big[\{\sup_{0\leq s\leq t} f(x_{s\wedge\tau_n^d})\}^{1/2} \{\int_0^{t} \sup_{0\leq s'\leq s} f(x_{s'\wedge\tau_n^d})ds\}^{1/2}\Big]$$

$$\leq \tfrac{1}{2} E^{\nu^{d,n}}\Big[\sup_{0\leq s\leq t} f(x_{s\wedge\tau_n^d})\Big] + c_2(\theta) \int_0^t E^{\nu^{d,n}}\Big[\sup_{0\leq s'\leq s} f(x_{s'\wedge\tau_n^d})\Big]ds$$

(4.10)

where the last step follows from the elementary inequality $\sqrt{ab} \leq \frac{a+b}{2}$ $(a, b > 0)$. The constant $c_2(\theta) = 160\,\theta$.

From (4.8),

$$\sup_{0\leq s\leq t} f(x_{s\wedge\tau_n^d}) \leq \sup_{0\leq s\leq t} |M^f_{s\wedge\tau_n^d}| + f(x_0) + (15\theta) \int_0^t \sup_{0\leq s\leq t} f(x_{s'\wedge\tau_n^d})ds\ .$$

Writing $c_3(\theta)$ for $c_2(\theta) + 15\theta$, taking expectations, using (4.10) and (IC_d) we obtain the inequality

$$E^{\nu^{d,n}}\Big[\sup_{0\leq s\leq t} f(x_{s\wedge\tau_n^d})\Big] \leq 2\,c_0 + 2\,c_3(\theta) \int_0^t E^{\nu^{d,n}}\Big[\sup_{0\leq s'\leq s} f(x_{s'\wedge\tau_n^d})\Big]ds$$

Another application of Gronwall's inequality yields the uniform bound

$$E^{\nu^{d,n}}\Big[\sup_{0\leq s\leq t} f(x_{s\wedge\tau_n^d})\Big] \leq 2\,c_0\, e^{2\,c_3(\theta)T} \qquad (0 \leq t \leq T)\ . \qquad (4.11)$$

Fix $R > 0$. Then for $n > R$,

$$\nu^{d,n}\Big[\sup_{0\leq s\leq t} |x_s| > R\Big] \leq \nu^{d,n}\Big[\sup_{0\leq s\leq t} |x_{s\wedge\tau_n^d}| > R\Big]$$

$$\leq E^{\nu^{d,n}}\left[\sup_{0\leq s\leq t} f(x_{s\wedge\tau_n^d})\right] / f(R)$$

$$\leq \frac{2\, c_0\, e^{2c_3(\theta)T}}{(1+R^2)\{\log(3+R^2)\}^2} . \qquad (4.12)$$

Now, if $\epsilon > 0$, choosing $\bar{R}$ such that the right hand side of the inequality (4.12) is less than ϵ, we have

$$\nu^{d,n}\left[\sup_{0\leq s\leq t} |x_s| > \bar{R}\right] < \epsilon \qquad (4.13)$$

for all $n > \bar{R}$.

Using (4.12) and taking f_1 to be the function introduced in the statement of the theorem the following inequality is obtained:

$$E^{\nu^{d,n}}\left[\sup_{0\leq s\leq t} f_1(x_s)\right] \leq c_4(\theta;T) < \infty . \qquad (4.14)$$

The bound being independent of n and d. Details of the proof will be given a little later when a similar inequality is proved for ν^d.

Next, let τ_R be the stopping time

$$\tau_R(x) := \inf\{t \geq 0 : |x_t| \geq R \quad \text{or} \quad t > T\} .$$

The continuous function $|b_t^d(\xi)b_t^{d*}(\xi)|$ bounded on the compact set $\{(t,\xi) : 0 \leq t \leq T, |\xi| \leq R\}$ by some number β_R which is independent of d in view of the uniformity of the bounds in (LG_d)(b) and (CC_d). The stopping time τ_R is also bounded, so by Doob's optional sampling theorem (applied at $\tau_R \leq \tau_n$), the $\mathbb{R}^d$-valued process

$$M_t^R(x) := x_{t\wedge\tau_R} - \int_0^{t\wedge\tau_R} a_s(x_s)ds$$

is a martingale for each $\nu^{d,n}$ with $n \geq R$, with bracket process

$$\langle M^R\rangle_t = \int_0^{t\wedge\tau_R} b_s^d(x_s)b_s^{d*}(x_s)ds$$

satisfying

$$\mathrm{tr}(\langle M^R\rangle_{t_2} - \langle M^R\rangle_{t_1}) \leq (t_2-t_1) \sup_{s\leq T\wedge\tau_R} |b_s^d(x_s)b_s^{d*}(x_s)| \leq \beta_R(t_2-t_1) .$$

By Lemma 4.1 and (LG_d)(a) there is a number δ such that for all $n \geq R$,

$$\nu^{d,n}[x : \sup_{\substack{0<s<\tau\leq T\\ t-s<\delta}} |M_t^R - M_s^R| > \epsilon/2] < \epsilon/2,$$

$$\left|\int_{s\wedge\tau_R}^{t\wedge\tau_R} a_s(x_s)ds\right| < \delta\sqrt{c_d(1+R^2)} \leq \epsilon/2$$

and hence

$$\nu^{d,n}[x : \sup_{\substack{0<s<t\leq T\\ t-s<\delta}} |x_t-x_s| > \epsilon] \leq \nu^{d,n}[x : \sup_{0<s<t\leq T} |x_s| > R]$$

$$+ \nu^{d,n}[x : \sup_{\substack{0<s<t\leq T\\ t-s<\delta}} |M_t^R - M_s^R| > \epsilon/2] < \epsilon.$$

By Theorem 8.2 of [1], $\{\nu^{d,n}\}$ is a tight sequence and so has a cluster point ν^d. Using the fact that $\nu^{d,n}$ satisfies the martingale problem for $a_t^{d,n}$ and $b_t^{d,n}$, the uniform bound (4.14) and the tightness of $\{\nu^{d,n}\}$ we can show (after a routine argument) that M_t^f in (4.2) ($f \in C_b^2$) is a

$$(C_{\mathbb{R}^d}, \underline{B}_{C_{\mathbb{R}^d}}, \underline{B}^t_{C_{\mathbb{R}^d}}, \nu^d)$$

continuous ν^d-martingale. This proves the first assertion of the theorem. It remains to show (4.3) and (4.4). From (4.12) taking, without loss of

generality that $\nu^{d,n} \Rightarrow \nu^d$, it follows that

$$\nu^d[\sup_{0\le s\le t} |x_s| > R] \le \frac{2c_0\, e^{2c_3(\theta)T}}{(1+R^2)\{\log(3+R^2)\}^2}$$

which is (4.3).

Now let us write $f_1(a) = (1 + a^2) \log \log(3 + a^2)$. Since f_1 is an increasing function for $a \ge 0$, by a standard formula we obtain

$$\begin{aligned} E^{\nu^d}[\sup_{0\le s\le t} f_1(x_s)] &= \int_0^\infty \nu^d[\sup_{0\le s\le t} f_1(x_s) > y]dy \\ &\le \int_0^\infty \nu^d[\sup_{0\le s\le t} |x_s| > a]\, f_1'(a)da\ . \\ &\le 2c_0\, e^{2c_3(\theta)T} \int_0^\infty \frac{f_1'(a)}{f(a)} da\ . \end{aligned}$$

The last integral is the sum of two integrals. The integrand in the first is of the order $\dfrac{\log\, \log(3+a^2)}{a\{\log(3+a^2)\}^2}$ as $a \to \infty$; in the second, the integrand is of the order $\dfrac{1}{a[\log(3+a^2)]^3}$. Hence $\int_0^\infty \dfrac{f_1'(a)}{f(a)} da < \infty$ and (4.4) is proved. ∎

Remark. The uniformity of the bound in (4.4) w.r.t d is of crucial importance and has been derived here with an eye on Theorem 5.2 which treats the corresponding infinite dimensional problem. It may be noted that in the proof of the above theorem, the uniformity with respect to n of the bound (4.14) is used in showing that ν^d is a martingale solution of the finite dimensional problem.

5. Martingale Problems in Nuclear Spaces. In this section we construct and solve finite-dimensional approximations to the stochastic differential equation (1.1) and prove that the solutions converge weakly along a subsequence to a solution to the martingale problem in Φ'. This method of finding a solution to the infinite-dimensional problem is patterned after the "Galerkin" method of reducing the problem of solving parabolic partial differential equations to that of solving finite systems of ordinary differential equations.

Fix a continuous positive quadratic form $Q[\cdot,\cdot]$ on $\Phi \times \Phi$, a Borel probability measure μ_0 on Φ', and continuous functions $A : \mathbb{R}_+ \times \Phi'$ and $B : \mathbb{R}_+ \times \Phi' \to L(\Phi';\Phi')$ all satisfying conditions (IC), (CC), (LG), and (JC) of (3.3). Note that $Q[\varphi,\varphi] = (Q_r^{1/2}\varphi, Q_r^{1/2}\varphi)_r = \sum_{i=1}^{\infty} (Q_r^{1/2})^* h_i^{-r}[\varphi]^2$, where Q_r is a positive selfadjoint nuclear operator on some H^r and * means the adjoint operator with respect to the dual pair $u[\varphi]$ on $\Phi' \times \Phi$. Fix any $T > 0$ and let $p > m \geq r$ be the indices such that the injection from H^p to H^m is Hilbert-Schmidt and $\theta > 0$ is the constant appearing in (3.3).

For each integer $d \geq 1$ and $\xi \in \mathbb{R}^d$ set $u = J_d^{-m}\xi := \Sigma_{i\leq d}\, \xi_i h_i^{-m}$ and define a vector-valued function $a^d : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and a nonnegative-definite matrix-valued function $b^d : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ by

$$a_t^d(\xi)_j := A_t(u)[h_j^m], \quad \text{and}$$

$$b_t^d(\xi)_{ij} = (Q_r^{1/2})^* h_j^{-r}[B_t^*(u)h_i^m] .$$

Since A and B satisfy (CC) and (LG), we can verify

$$(\xi,\, a_t^d(\xi)) = \sum_{i\leq d} \xi_i\, A_t(u)[h_i^m] = A_t(u)[J_{-m}u],$$

$$|a_t^d(\xi)|^2 \leqq \sum_{i\leqq d} |A_t(u)|_{-p}^2\, |h_i^m|_p^2$$

$$\leqq \sum_{i\leq d} \theta(1 + |u|^2_{-m}) \; |h^m_i|^2_p$$

$$= c_d(1 + |u|^s_{-m}) = c_d(1 + |\xi|^2)$$

and $|b^d_t(\xi)b^{d*}_t(\xi)| = \mathrm{tr}\; b^d_t(\xi)b^{d*}_t(\xi)$

$$= \sum_{i\leqq d} \sum_{j\leqq d} (Q^{1/2}_r)^* \, h^{-r}_j [B^*_t(u)h^m_i]^2$$

$$\leq \sum_{i\leqq d} \sum_{j<\infty} (Q^{1/2}_r)^* \, h^{-r}_j [B^*_t(u)h^m_i]^2$$

$$= \sum_{i\leqq d} Q[B^*_t(u)h^m_i, \; B^*_t(u)h^m_i]$$

$$\leqq |Q_{B_t}|u)|_{-m,-m} \quad .$$

It follows that a^d_t and b^d_t satisfy the finite dimensional bounds (for all $t \leq T$ and $\xi \in \mathbb{R}^d$)

$$2(\xi, a^d_t(\xi)) + |b^d_t(\xi)b^{d*}_t(\xi)| \leq \theta(1 + |\xi|^2) \qquad (\mathcal{C}_d)$$

$$|b^d_t(\xi)b^{d*}_t(\xi)| \leq \theta(1 + |\xi|^2) \qquad (LG_d)$$

for fixed T, θ independent of d and $c_d = \theta \sum_{i\leq d} |h^m_i|^2_p$. Here we remark the following for later use.

$$\sum_{i\leqq d_0} |a^d_t(\xi)_i|^2 \leqq c_{d_0}(1 + |\xi|^2) \qquad \text{if } d_0 \leq d, \qquad (5.1)$$

where $c_{d_0} = \sum_{i\leqq d_0} \theta|h^m_i|^2_p$ is independent of d.

Furthermore, the measure on μ_0 on Φ' induces on each $\mathbb{R}^d$ a measure $\nu^d_0 := \mu_0 \circ (\Pi^{-m}_d)^{-1} \circ J^{-m}_d$ satisfying the bound

$$\int_{\mathbb{R}^d} (1 + |\xi|^2)[\log(3 + |\xi|^2)]^2 \, \nu_0^d(d\xi)$$

$$= \int_{\Phi'} (1 + |\Pi_d^{-m} u|_{-m}^2)[\log(3 + (\Pi_d^{-m} u|_{-m}^2)]^2 \mu_0(du)$$

$$\leqq \int_{\Phi'} (1 + |u|_{-m}^2)[\log(3 + |u|_{-m}^2)]^2 \, \mu_0(du) = c_0 < \infty, \qquad (IC_d)$$

again uniformly in the dimension d. It follows from Theorem 4.1 that there exists on $(C_{\mathbb{R}^d}, \underline{B}_{C_{\mathbb{R}^d}})$ a measure v^d satisfying the initial condition $\nu^d \circ x_0^{-1} = \nu_0^d$ and solving the martingale problem for L_s, so that

$$(M_t^d)_j = (x_t^d)_j - (x_o^d)_j - \int_0^t a_s^d(x_s^d)_j ds \qquad (5.2)$$

is a $(C_{\mathbb{R}^d}, \underline{B}_{C_{\mathbb{R}^d}}, \underline{B}^t_{C_{\mathbb{R}^d}}, \nu^d)$ continuous L^2 martingale for each $j = 1,2,\dots,d$.

Define a mapping j_d^{-m} from $C_{\mathbb{R}^d}$ to $C_{H_d^{-m}}$ by

$$(j_d^{-m} x^d)(t) := J_d^{-m} x^d(t).$$

Then each such measure ν^d induces a measure μ^d on $(C_{\Phi'}, \underline{B}_{C_{\Phi'}})$ with support in $(C_{H^{-m}}, \underline{B}_{C_{H^{-m}}})$ via the relation

$$\mu^d[A] = \nu^d[(j_d^{-m})^{-1}(A \cap C_{H_d^{-m}})], \; A \in \underline{B}_{C_{\Phi'}}. \qquad (5.3)$$

Theorem 5.1. For any continuous coefficient functions $A : \mathbb{R}_+ \times \Phi' \to \Phi'$ and $B : \mathbb{R}_+ \times \Phi' \to L(\Phi';\Phi')$ satisfying conditions (CC) and (LG) of (3.3), and any measure μ_0 on $\underline{B}(\Phi')$ satisfying (IC), the family (μ^d) defined above is tight on $(C_{H^{-p}}, \underline{B}_{C_{H^{-p}}})$ and also on $(C_{\Phi'}, \underline{B}_{C_{\Phi'}})$. Here $p > m$ is the index appearing in (L).

Proof. Again we apply Theorem 8.2 of [1]. For any $\epsilon > 0$, choose $R > 0$ large

enough to insure that

$$\frac{2c_0 e^{2c_3(\theta)T}}{(1+R^2)[\log(3+R^2)]^2} < \epsilon/2.$$

By (4.3) we have

$$\mu^d[x \in C_{\Phi'} : \sup_{0 \leq t \leq T} |x_t|_{-m} > R] = \nu^d[x^d \in C_{\mathbb{R}^d} : \sup_{0 \leq t \leq T} |x_t^d| > R] < \epsilon/2 . \tag{5.4}$$

Since $\sum_{j=1}^{\infty} |h_j^p|_m^2 < \infty$, we can choose some d_0 such that

$$(\sum_{j=d_0+1}^{\infty} x[h_j^p]^2)^{\frac{1}{2}} \leqq \epsilon/8 \qquad \text{if} \quad |x|_{-m} \leqq R. \tag{5.5}$$

From the way of choosing the CONSs $\{h_j^m\}$ and $\{h_j^p\}$, we get

$$(\sum_{j=1}^{d_0} (x-y)[h_j^p]^2)^{\frac{1}{2}} \leqq \alpha_0(\sum_{j=1}^{d_0} (x-y)[h_j^m]^2)^{\frac{1}{2}} \quad \text{if} \quad x,y \in H^{-m},$$

where α_0 is a constant only depending on d_0, m and p.

On the other hand, in a manner similar to that in Section 4, the L^2-martingale

$$M_t^R = x_{t\wedge\tau_R}^d - \int_0^{t\wedge\tau_R} a_s(x_s^d)ds$$

satisfies

$$\nu^d[x^d : \sup_{\substack{0 \leq s \leq t \leq T \\ |s-t| \leq \eta}} |M_t^R - M_s^R| > \frac{\epsilon}{2\alpha_0}] < \epsilon/2 \quad \text{if} \quad \eta \leq \delta_1, \tag{5.6}$$

where δ_1 is some constant independent of d (by Lemma 4.1).

From now on suppose $d \geqq d_0$. Since

$$\sum_{j=1}^{d_0} (a_s^d(x_s^d)_j)^2 \leqq c_{d_0}(1 + |x_s^d|^2)$$

from (5.1), if $\sup_{0\leq t\leq T} |x_t^d| \leq R$, we have some $\delta < \delta_1$, (δ independent of d), such that for $|t-s| \leq \delta$ and t, s $\in$ [0,T],

$$\left(\sum_{j=1}^{d_0} \left(\int_s^t a_\tau^d(x_\tau^d)_j d\tau\right)^2\right)^{1/2} \leqq \epsilon/4\ \alpha_0 . \tag{5.7}$$

Since

$$M_t^d := x_t^d - \int_0^t a_s^d(x_s^d)ds = M_t^R$$

if $\sup_{0\leq t\leq T} |x_t^d| \leqq R$, the inequalities (5.4), (5.5), (5.6) and (5.7) yield

$$\mu^d[x: \sup_{\substack{0\leq s\leq t\leq T\\ |s-t|\leq\delta}} |x_t-x_s|_{-p} > \epsilon] \tag{5.8}$$

$$\leqq \mu^d[x: \sup_{0\leq s\leq t\leq T} |x_t|_{-m} > R]$$

$$+ \mu^d[x: \sup_{0\leq t\leq T} |x_t|_{-m} \leqq R, \sup_{\substack{0\leq s\leq t\leq T\\ |s-t|\leq\delta}} |x_t-x_s|_{-p} > \epsilon]$$

$$< \epsilon/2 + \nu^d[x^d: \sup_{0\leq t\leq T} |x_t^d| \leqq R, \sup_{\substack{0\leq s\leq t\leq T\\ |s-t|\leq\delta}} \left(\sum_{j=1}^{d_0} ((x_t^d)_j - (x_s^d)_j)^2\right)^{1/2} > 3\ \epsilon/4\ \alpha_0]$$

$$\leqq \epsilon/2 + \nu^d[x^d: \sup_{0\leq t\leq T} |x_t^d| \leqq R, \sup_{\substack{0\leq s\leq t\leq T\\ |s-t|\leq\delta}} \left(\sum_{j=1}^{d_0} ((M_t^d)_j - (M_s^d)_j)^2\right)^{1/2} > \epsilon/2\ \alpha_0]$$

$$\leqq \epsilon/2 + \nu^d[x^d: \sup_{0\leq t\leq T} |x_t^d| \leqq R, \sup_{\substack{0\leq s\leq t\leq T\\ |s-t|\leq\delta}} |M_t^d - M_s^d| > \epsilon/2\ \alpha_0]$$

$$\leqq \epsilon/2 + \upsilon^d[x^d: \sup_{0 \le t \le T} |x_t^d| \leqq R, \sup_{\substack{0 \le s \le t \le T \\ |s-t| \le \delta}} |M_t^R - M_s^R| > \epsilon/2\, \alpha_0]$$

$$< \epsilon .$$

Since the ball $\{x; |x|_{-m} \leqq R\}$ is compact in H^{-p}, by (5.8) and the Hilbert space analogue of Theorem 8.2 in [1], (μ^d) is tight as a family of measures on the space $C_{H^{-p}}$. Since the embedding of $C_{H^{-p}}$ in C_Φ, is continuous, compact sets in $C_{H^{-p}}$ are also compact in C_Φ, and the theorem is proved.

Just before Theorem 5.2, we need another lemma guaranteeing the uniform integrability to be used later.

Lemma 5.1. For $0 \le t \le T$,

(1) $$E^{\mu^d}[\sup_{0 \le s \le t} (1 + |x_s|_{-m}^2) \log\log(3 + |x_s|_{-m}^2)] \le c_4(\theta), \tag{5.9a}$$

(2) $$E^{\mu^*}[\sup_{0 \le s \le t} (1 + |x_s|_{-m}^2) \log\log(3 + |x_s|_{-m}^2)] \le c_4(\theta), \tag{5.9b}$$

(3) $$E^{\mu^*}[1 + |x_t|_{-m}^2] \le c_5(\theta), \tag{5.9c}$$

(4) $$\mu^*[\sup_{0 \le s \le t} |x_s|_{-m} > R] \leqq \frac{2\, c_0\, e^{2c_3(\theta)T}}{(1+R^2)[\log(3+R^2)]^2} , \tag{5.9d}$$

where $c_4(\theta)$ is the constant appearing in (4.4) and $c_5(\theta)$ is also a constant independent of d.

In proving (5.9b), (5.9c) and the following Theorem 5.2, we use frequently the Skorohod imbedding theorem guaranteed by Theorem 5.1 such that there exist H^{-p}-valued processes X_t^d with the distribution μ^d and X_t with the distribution μ^* on a probability space $(\Omega, \mathcal{F}, P)$ and further X_t^d converges to X_t almost surely in $C_{H^{-p}}$.

Proof. The inequality (5.9a) follows from (4.4). Since $\sup_{0\leq s\leq t}(1+|x_s|_{-m}^2)\log\log(3+|x_s|_{-m}^2)$ is lower semi-continuous on $C_{\Phi'}$, by (5.9a) we get

$$E^{\mu^*}[\sup_{0\leq s\leq t}(1+|x_s|_{-m}^2)\log\log(3+|x_s|_{-m}^2)]$$

$$= E[\sup_{0\leq s\leq t}(1+|X_s|_{-m}^2)\log\log(3+|X_s|_{-m}^2)]$$

$$\leqq \liminf_{d\to\infty} E[\sup_{0\leq s\leq t}(1+|X_s^d|_{-m}^2)\log\log(3+|X_s^d|_{-m}^2)]$$

$$= \liminf_{d\to\infty} E^{\mu^d}[\sup_{0\leq s\leq t}(1+|x_s|_{-m}^2)\log\log(3+|x_s|_{-m}^2)]$$

$$\leqq c_4(\theta) .$$

The bound (5.9c) can be obtained similarly from

$$E^{\mu^d}[1+|x_t|_{-m}^2] \leqq c_5(\theta), \tag{5.10a}$$

which follows easily from (5.9a).

Also by (4.3) we get

$$\mu^d[\sup_{0\leq s\leq t}|x_s|_{-m} > R] \tag{5.10b}$$

$$= \nu^d[\sup_{0\leq s\leq t}|x_t^d| > R]$$

$$\leqq \frac{2c_0\, e^{2c_3(\theta)T}}{(1+R^2)[\log(3+R^2)]^2}$$

and hence noting that $\sup_{0\leq s\leq t}|x_t|_{-m}$ is lower semi-continuous with respect to x in $C_{\Phi'}$, we have

$$\mu^*[\sup_{0\leq s\leq t}|x_s|_{-m} > R]$$

$$\leqq \liminf_{d\to\infty} \mu^d[\sup_{0\leq s\leq t} |x_s|_{-m} > R]$$

$$\leqq \frac{2c_0\ e^{2c_3(\theta)T}}{(1+R^2)[\log(3+R^2)]^2}\ ,$$

since $\{\sup_{0\leq s\leq t} |x_s|_{-m} > R\}$ is open in $C_{\Phi'}$.

Theorem 5.2. Let A, B, Q, and μ_0 satisfy the hypotheses (IC), (CC), (LG), (JC) of (3.3). Then any cluster point μ^* of the tight family (μ^d) solves the martingale problem for $0 \leq t \leq T$ on $C_{\Phi'}$ with initial distribution μ_0 and generator

$$L_t\ f(x) = \tilde{f}'(x[\varphi])A_t(x)[\varphi] + ½\ \tilde{f}''(x[\varphi])Q(B_t^*(x)\varphi,\ B_t^*(x)\varphi),$$

$$f(x) = \tilde{f}(x[\varphi]) \in \mathfrak{D}_b^2(\Phi')\ .$$

Furthermore, any such measure μ^* satisfies the inequalities

$$E^{\mu^*}[\sup_{0\leq s\leq t} (1 + |x_s|_{-m}^2) \log\log(3 + |x_s|_{-m}^2)] \leqq c_4(\theta)\ ,$$

$$\mu^*[\sup_{0\leq s\leq t} |x_s|_{-m} > R] \leqq \frac{2\ c_0\ e^{2c_3(\theta)T}}{(1+R^2)[\log(3+R^2)]^2}\ ,$$

and

$$\mu^*[\sup_{0\leq s\leq t} |x_s|_{-m} < \infty] = 1\ .$$

Proof. By Theorem 5.1, we may assume that μ^d converges weakly to μ^* without loss of generality. Since $\mu_0(H^{-m}) = 1$, we have

$$\int_{C_{\Phi'}} f(x_0)\mu^*(dx) = \lim_{d\to\infty} \int_{C_{\Phi'}} f(x_0)\mu^d(dx) = \lim_{d\to\infty} \int_{H^{-m}} f(\Pi_d^{-m}u)\ \mu_0(du) \qquad (5.11)$$

$$= \int_{\Phi'} f(u)\mu_0(du)$$

for any bounded continuous function f on Φ'. Noting that $\mu^* \circ x_0^{-1}$ and μ_0 are Radon measures on Φ', by the monotone class theorem in the form of Theorem 1.21 of [2] and (5.11) we get $\mu^* \circ x_0^{-1} = \mu_0$.

Now we must verify that for each $0 \leq t \leq t' \leq T$ and bounded $\underline{B}^t_{C_{\Phi'}}$ measurable function g,

$$E^{\mu^*}[g(x)(M^\varphi_{t'}(x) - M^\varphi_t(x))] = 0 \tag{5.12}$$

and the sharp bracket function

$$\langle M^\varphi_t \rangle = \int_0^t Q(B^*_s(x_s)\varphi, B^*_s(x_s)\varphi)ds, \tag{5.13}$$

where

$$M^\varphi_t(x) = x_t(\varphi) - x_0(\varphi) - \int_0^t A_s(x_s)[\varphi]ds .$$

We will first verify (5.11) for bounded, continuous functions g which depend on x at only finitely many times, then extend to larger classes of g (by the monotone class theorem). Suppose

$$g(x) = \tilde{g}(x_{t_1}, \ldots, x_{t_m})$$

for some $N \in \mathbb{N}$, $0 \leq t_1 < t_2 < \ldots < t_N \leq t$, and $\tilde{g} \in C_b(\Phi'^N)$. To prove (5.12), we will derive some estimates. By (5.9d) and (5.10b), we get $\mu^d(x; |x_s|_{-m} < \infty) = 1$ and $\mu^*(x; |x_s|_{-m} < \infty) = 1$ and hence from (LG), (5.10a) and (5.9c), we have

$$E^{\mu^d}[|A_s(x_s)[\varphi]|^2] \tag{5.14}$$

$$\leqq E^{\mu^d}[|A_s(x_s)|^2_{-p}|\varphi|^2_p]$$

$$\leqq E^{\mu^d}[\theta|\varphi|_p^2(1 + |x_s|_{-m}^2)]$$

$$\leqq \theta\ c_5(\theta)|\varphi|_p^2$$

and

$$E^{\mu^*}[|A_s(x_s)[\varphi]|^2] \tag{5.15}$$

$$\leqq \theta\ c_5(\theta)|\varphi|_p^2 .$$

By assumption (JC) (i), if $x \in H^{-m}$,

$$Q[B_s^*(x)\varphi,\ B_s^*(x)\varphi]$$

$$= \sum_{k=1}^{\infty} B_s(x)(Q_r^{1/2})^* h_k^{-r}[\varphi]^2$$

$$\leqq |Q_{B_s(x)}|_{-m,-m}|\varphi|_m^2$$

and hence we get similarly

$$E^{\mu^d}[Q[B_s^*(x_s)\varphi,\ B_s^*(x_s)\varphi]] \tag{5.16}$$

$$\leqq E^{\mu^d}[|\varphi|_m^2|\ Q_{B_s(x_s)}|_{-m,-m}]$$

$$\leqq |\varphi|_m^2\ E^{\mu^d}[\theta(1+|x_s|_{-m}^2)]$$

$$\leqq \theta\ c_5(\theta)|\varphi|_m^2 .$$

and

$$E^{\mu^*}[Q[B_s^*(x_s)\varphi,\ B_s^*(x_s)\varphi]] \leqq \theta\ c_5(\theta)|\varphi|_m^2 . \tag{5.17}$$

Set $\varphi^d = \sum_{j=1}^{d} (\varphi, h_j^p)_p h_j^p$ and $\varphi_d = \varphi - \varphi^d$. Since $|\cdot|_{-p} \leqq |\cdot|_{-m}$ and g is bounded, by (5.9c) and (5.15), for any $\epsilon > 0$, we have some N such that

$$|E^{\mu^*}[g(x)\ M_t^{\varphi}(x)] - E^{\mu^*}[g(x)\ M_t^{\varphi N}(x)]| < \epsilon/2$$

and hence

$$\begin{aligned} |E^{\mu^*}&[g(x)(M_{t'}^{\varphi}(x) - M_t^{\varphi}(x))] \\ &- E^{\mu^*}[g(x)(M_{t'}^{\varphi N}(x) - M_t^{\varphi N}(x))]| < \epsilon. \end{aligned} \tag{5.18}$$

On the other hand, since

$$\varphi^N = \sum_{j=1}^{n(N)} \alpha_j\ h_j^m + \theta_N, \quad \|\theta_N\|_m = 0,$$

we get

$$J_d^{-m}\ x_s^d[\varphi^N] = \sum_{j=1}^{N} \alpha_j (x_s^d)_j \qquad \text{if} \quad d \geq N$$

and since by assumption $\|\theta_N\|_p = 0$,

$$M_t^{\varphi N}(J_d^{-m} x^d) = \bar{M}_t^d := \sum_{j=1}^{N} \alpha_j \{(x_t^d)_j - (x_0^d)_j - \int_0^t a_s^d (x_s^d)_j ds\} .$$

Noting that M_t^d is a martingale if $d \geqq N$ by (5.2) and using the boundedness of g, the uniform integrability in (5.15) and the Skorohod theorem, we have

$$\begin{aligned} E^{\mu^*}&[g(x)\ (M_{t'}^{\varphi N}(x) - M_t^{\varphi N}(x))] \\ &= E[g(X)(M_{t'}^{\varphi N}(X) - M_t^{\varphi N}(X))] \\ &= \lim_{d \to \infty} E[g(X^d)\ (M_{t'}^{\varphi N}(X^d) - M_t^{\varphi N}(X^d))] \end{aligned}$$

$$= \lim_{d\to\infty} E^{\mu^d}[g(x)\ (M_{t'}^{\varphi N}(x) - M_t^{\varphi N}(x))]$$

$$= \lim_{d\to\infty} E^{\nu^d}[g(J_d^{-m}\ x^d)(\bar{M}_{t'}^d - \bar{M}_t^d)] = 0\ ,$$

which, together with (5.18), implies (5.12) holds for the special class of functions g mentioned before. Since Φ' is a standard space, by the monotone class theorem (again in the form of Theorem 1.21 of [2]) (5.12) holds for all bounded $\underline{B}^t_{C_{\Phi'}}$ - measurable functions g.

To prove (5.13), we first choose the following sufficiently large N. Since $|\cdot|_m \leqq |\cdot|_p$,

$$\lim_{d\to\infty} |\varphi-\varphi^d|_m = 0 \tag{5.19}$$

and hence

$$|\varphi^d|_m \le |\varphi|_m + 1 \qquad \text{if} \quad d \geqq d_0\ . \tag{5.20}$$

By (5.9c), (5.15), (5.17), (5.19) and (5.20), for any $\epsilon > 0$ we have some $N \geqq d_0$ such that

$$|E^{\mu^*}[\int_0^t Q[B_s^*(x_s)\varphi^N,\ B_s^*(x_s)\varphi^N]ds - \int_0^t Q[B_s^*(x_s)\varphi,\ B_s^*(x_s)\varphi]ds]| < \epsilon/2 \tag{5.21}$$

and

$$|E^{\mu^*}[M_t^{\varphi}(x)^2] - E^{\mu^*}[M_t^{\varphi N}(x)^2]| < \epsilon/2\ . \tag{5.22}$$

Since $M_t^{\varphi N}(x)$ is continuous on $C_{\Phi'}$, by the Skorohod theorem and the uniform integrability in (5.9a), we have

$$E^{\mu^*}[M_{t'}^{\varphi N}(x)^2] = E[M_t^{\varphi N}(X)^2] \tag{5.23}$$

$$= \lim_{d\to\infty} E[M_t^{\varphi N}(X^d)^2]$$

$$= \lim_{d\to\infty} E^{\mu^d}[M_t^{\varphi N}(x)^2]$$

$$= \lim_{d\to\infty} E^{\nu^d}[M_t^{\varphi N}(J_d^{-m}\, x^d)^2]$$

$$= \lim_{d\to\infty} E^{\nu^d}[\int_0^t \sum_{k=1}^{d} B_s(J_d^{-m}\, x_s^d)(Q_r^{1/2})^* \, h_k^{-r}[\varphi^N]^2 \, ds]$$

$$= \lim_{d\to\infty} E^{\mu^d}[\int_0^t \sum_{k=1}^{d} B_s(x_s)(Q_r^{1/2})^* \, h_k^{-r}[\varphi^N]^2 \, ds]$$

$$= \lim_{d\to\infty} E[\int_0^t \sum_{k=1}^{d} B_s(X_s^d)(Q_r^{1/2})^* \, h_k^{-r}[\varphi^N]^2 \, ds] .$$

Now for $\omega \in \{\omega;\ |X_s^d|_{-m} < \infty\}$, we get

$$\sum_{k=1}^{d} B_s(X_s^d)(Q_r^{1/2})^* \, h_k^{-r}[\varphi^N]^2 \tag{5.24}$$

$$\leqq \sum_{k=1}^{\infty} B_s(X_s^d)(Q_r^{1/2})^* \, h_k^{-r}[\varphi^N]^2 = Q[B_s^*(X_s^d)\varphi^N,\ B_s^*(X_s^d)\varphi^N]$$

$$\leqq |Q_{B_s(X_s^d)}|_{-m,-m} \, |\varphi^N|_m^2$$

$$\leqq \theta(1 + |X_s^d|_{-m}^2)\,|\varphi^N|_m^2 .$$

Since $P(|X_s^d|_{-m} < \infty) = 1$ from (5.10b), noticing the uniform integrability (5.9a) and the continuity of $Q[B_s^*(\cdot)\varphi,\ B_s^*(\cdot)\varphi]$ on Φ' by (JC)(ii) the right hand side of (5.23) is dominated by

$$\lim_{d\to\infty} E[\int_0^t Q[B_s^*(X_s^d)\varphi^N,\ B_s^*(X_s^d)\varphi^N]ds] \tag{5.25}$$

$$= E[\int_0^t Q[B_s^*(X_s)\varphi^N,\ B_s^*(X_s)\varphi^N]\ ds]\ .$$

On the other hand, if $d \geqq n$,

$$\sum_{k=1}^{d} B_s(X_s^d)(Q_r^{1/2})^* h_k^{-r}[\varphi^N]^2 \tag{5.26}$$

$$\geqq \sum_{k=1}^{n} B_s(X_s^d)(Q_r^{1/2})^* h_k^{-r}[\varphi^N]^2$$

so that we have

$$\liminf_{d\to\infty} \sum_{k=1}^{d} B_s(X_s^d)(Q_r^{1/2})^* h_k^{-r}[\varphi^N]^2 \tag{5.27}$$

$$\geqq \sum_{k=1}^{n} \liminf_{d\to\infty} B_s(X_s^d)(Q_r^{1/2})^* h_k^{-r}[\varphi^N]^2$$

$$= \sum_{k=1}^{n} B_s(X_s)(Q_r^{1/2})^* h_k^{-r}[\varphi^N]^2 .$$

The inequalities (5.26) and (5.27) yield

$$\liminf_{d\to\infty} \sum_{k=1}^{d} B_s(X_s^d)(Q_r^{1/2})^* h_k^{-r}[\varphi^N]^2 \tag{5.28}$$

$$\geqq \sum_{k=1}^{\infty} B_s(X_s)(Q_r^{1/2})^* h_k^{-r}[\varphi^N]^2$$

$$= Q[B_s^*(X_s)\varphi^N,\ B_s^*(X_s)\varphi^N]\ .$$

By (5.28) and Fatou's lemma, the right hand side of (5.23) is larger than

$$E[\int_0^t Q[B_s^*(X_s)\varphi^N,\ B_s^*(X_s)\varphi^N] ds],$$

which gives

$$E^{\mu^*}[M_t^{\varphi N}(x)^2] = E[\int_0^t Q[B_s^*(X_s)\varphi^N, B_s^*(X_s)\varphi^N]ds]. \tag{5.29}$$

Summing up (5.21), (5.22), (5.23), and (5.29), we obtain (5.12), which completes the proof of Theorem 5.2. ∎

6. Existence of a Weak Solution. Let Ω be the canonical space $C_{\Phi'} = C(\mathbb{R}_+; \Phi')$ with filtration $\underline{F}_t = \underline{B}^t_{C_{\Phi'}}$, $0 \leq t < \infty$ and measure $P = \mu^*$, and consider the coordinate process $x_t(\omega) := \omega(t)$ for $\omega \in \Omega$. In Theorem 5.2 it has been shown that, for each $\varphi \in \Phi$, the real-valued process

$$M^\varphi_t = x_t[\varphi] - x_0[\varphi] - \int_0^t A_s(x_s)[\varphi]ds \tag{6.1}$$

is a continuous local martingale with sharp bracket function

$$\langle M^\varphi\rangle_t = \int_0^t Q[f^*_s\,\varphi,\ f^*_s\varphi]ds \tag{6.2}$$

where we set $f_s := B_s(x_s(\omega))$ and denote the adjoint of f_s by f^*_s, given by the relation

$$u[f^*_s\,\varphi] = (f_s u)[\varphi] \qquad u \in \Phi',\ \varphi \in \Phi\ .$$

and we also have

$$E[\int_0^T Q[f^*_s\,\varphi,\ f^*_s\,\varphi]ds] \leqq \theta\ c_5(\theta)T|\varphi|^2_m \tag{6.3}$$

for each $T > 0$. Repeating the argument in Section 2, we find a continuous H^{-p}-valued L^2-martingale M_t with $M_t[\varphi] = M^\varphi_t$ and operator-valued Meyer (or sharp bracket) process

$$A_t(\omega) = \langle M\rangle_t(\omega) = \sum_{j,k=1}^{\infty} \langle M^j, M^k\rangle_t(\omega)\ h^{-p}_j \otimes h^{-p}_k, \tag{6.4}$$

where we have abbreviated $M_t[h^{-p}_j]$ by M^j_t . An important consequence of (6.3) is

$$\sum_{j=1}^{\infty} Q[f^*_s(\omega)\ h^p_j,\ f^*_s(\omega)\ h^p_j]ds < \infty \tag{6.5}$$

a.s. for a.e. s, i.e. for all (s,ω) in a set $\Lambda \subset [0,T] \times \Omega$ with $\lambda \otimes P\ (\Lambda^c) = 0$,

where λ denotes Lebesgue measure. Note that, from (6.4), $\underline{A}_t \in H_{-p} \otimes H_{-p}$ a.s. and for g, $h \in H_{-p}$,

$$(\underline{A}_t(\omega)g,h)_{-p} = (\underline{A}_t(\omega),\ g \otimes h)_{-p,-p} = \sum_{j,k=1}^{\infty} \langle M^j, M^k\rangle_t(\omega)(g,h_j^{-p})_{-p}\ (h,h_k^{-p})_{-p}\ .$$

In particular,

$$(\underline{A}_t(\omega)h_j^{-p},\ h_k^{-p})_{-p} = \langle M^j, M^k\rangle_t(\omega) = \int_0^t Q[f_s^* h_j^p,\ f_s^* h_k^p]ds. \qquad (6.6)$$

From (6.5) and (6.6) it is seen that $\underline{A}_t(\omega)$ is a nuclear operator for almost every ω.

Let $(s,\omega) \in \Lambda$ such that $|x_s(\omega)|_{-m} < \infty$, $\tilde{h} \in H^p$ and find a sequence $\psi_n \in \Phi$ for which $|\psi_n - \tilde{h}|_p \to 0$. Setting $L_n^h(s,\omega) = \sum_{k=1}^{\infty} (Q_p^{1/2})^* h_k^{-p}[f_s^*\ \psi_n](h,h_k^{-p})_{-p}$ for $h \in H^{-p}$ and noting

$$Q[f_s^*(\omega)\varphi,\ f_s^*(\omega)\varphi] = \sum_{k=1}^{\infty} (Q_p^{1/2})^*\ h_k^{-p}[f_s^*(\omega)\varphi]^2$$

$$\leqq \Big(\sum_{k=1}^{\infty} Q[f_s^*(\omega)h_k^p,\ f_s^*(\omega)h_k^p]\Big)|\varphi|_p^2\ ,$$

we have

$$|L_n^h(w,\omega) - L_m^h(s,\omega)|^2 \to 0 \qquad \text{as } m,n \to \infty.$$

Since $P(\omega : |x_s|_{-m}^2 < \infty) = 1$, define

$$R_s(\omega)h[\tilde{h}] = \begin{cases} \lim\limits_{n\to\infty} L_n^h(s,\omega) & (s,\omega) \in \Lambda \cap \{\omega : |x_s|_{-m} < \infty\}, \\ 0, & \text{otherwise.} \end{cases}$$

Then there exists a fixed P-null set outside of which we have, for all $h \in H_{-p}$

$$(\underline{A}_t h,h)_{-p} = \int_0^t\ |R_s h|_{-p}^2\ ds, \qquad (6.7)$$

$$\sum_{j=1}^{\infty} |R_s(\omega)h_j^{-p}|_{-p}^2 = \sum_{j=1}^{\infty} Q[f_s^* h_j^p, f_s^* h_j^p] < \infty$$

and

$$|R_s(\omega)h|_{-p} \le |h|_{-p} \{\sum_{j=1}^{\infty} Q[f_s^*(\omega)h_j^p, f_s^*(\omega)h_j^p)\}^{1/2} < \infty \qquad s \in [0,t] .$$

We summarize its properties below:

Proposition 6.1.

(a) $R_s(\omega) : H_{-p} \to H_{-p}$ is a Hilbert-Schmidt operator;

(b) $R_s(\omega)h[\varphi] = \sum_{k=1}^{\infty} (Q_p^{1/2})^* h_k^{-p}[f_s^* \varphi] (h, h_k^{-p})_{-p}$.

(c) $(\underline{A}_t(\omega)g,h)_{-p} = \int_0^t (R_s(\omega)g, R_s(\omega)h)_{-p}\, ds$, a.s. .

Denoting by $R_s'(\omega)$ the H_{-p} adjoint of $R_s(\omega)$, we now show that for each $t \le T$,

$$\underline{A}_t(\omega) = \int_0^t R_s'(\omega)R_s(\omega)ds. \tag{6.8}$$

The operator integral on the right-hand-side is easily defined by noting that, for f and g in H_{-p}

$$\rho_t[g,h] := \int_0^t (R_s' R_s g,h)_{-p}\, ds$$

is a continuous bilinear form on H_{-p}, and hence

$$\begin{aligned}\rho_t[g,h] &= (\int_0^t R_s' R_s\, ds\; g,h)_{-p} \\ &= \int_0^t (R_s\, g, R_s h)_{-p}\, ds \\ &= (\underline{A}_t\, g,h)_{-p}\end{aligned}$$

from Proposition 6.1(c), and we have the representation (6.8).

Theorem 6.1. There exists a Φ'-valued Wiener process W_t such that the Φ'-martingale M_t has the representation

$$M_t = \int_0^t f_s \, dW_s \qquad 0 \leq t \leq T \tag{6.9}$$

Proof. It has already been shown that, over the interval [0,T], M_t is a continuous, H_{-p} valued, locally square-integrable martingale with bracket operator

$$\underline{A}_t(\omega) = \int_0^t R_s'(\omega) \, R_s(\omega) ds \ .$$

Now Lemmas IV.3.3, IV.3.4, and Theorem IV.3.5 of [13] apply to yield a CBM β_s on the Hilbert space H_{-p} adapted to the filtration $\underline{B}^t_{H^{-p}}$ and satisfying

$$M_t = \int_0^t \ R_s \, d\beta_s$$

Now we can define

$$W_t := \sum_{j=1}^{\infty} \beta_t^j (Q_p^{\frac{1}{2}})^* \, h_j^{-p} \ ,$$

where $\beta_t^j := \beta_t[h_j^p]$ for the chosen CBM β_s. It is easy to verify that W_t is a Φ'-valued Wiener process with covariance $E \, W_t[\varphi] \, W_s[\psi] = (s \wedge t) \, Q[\varphi, \psi]$. In fact, $W_t \in H_{-p}$ P-a.s. since

$$E[\,|W_t|_{-p}^2] = t \sum_{j=1}^{\infty} \, ((Q_p^{\frac{1}{2}})^* \, h_j^{-p}, \ (Q_p^{\frac{1}{2}})^* \, h_j^{\ -p})_{-p}$$

$$= t \sum_{j=1}^{\infty} Q[h_j^p, \ h_j^p] < \infty \ .$$

Since $\int_0^t R_s(\omega) d\beta_s[\varphi] = \sum_{k=1}^{\infty} \int_0^t R_s \ h_k^{-p}[\varphi] d\beta_s^k$ by the definition, it follows that,

for $t \leqq T$,

$$\int_0^t (f_s^* \varphi,\ h_j^p)_p \, dW_s[h_j^p] = \sum_{k=1}^{\infty} \int_0^t (f_s^* \varphi, h_j^p)_p \, (Q_p^{1/2})^* \, h_k^{-p}[h_j^p] d\, \beta_s^k,$$

$$\sum_{j=1}^{\infty} \int_0^t (f_s^* \varphi,\ h_j^q)_p \, dW_s[h_j^p] = \sum_{k=1}^{\infty} \int_0^t \sum_{j=1}^{\infty} (f_s^* \varphi, h_j^p)_p \, (Q_p^{1/2})^* \, h_k^{-p}[h_j^p] d\, \beta_s^k,$$

$$= \sum_{k=1}^{\infty} \int_0^t (Q_p^{1/2})^* \, h_k^{-p}[f_s^* \varphi] d\, \beta_s^k,$$

$$= \sum_{k=1}^{\infty} \int_0^t R_s \, h_k^{-p}[\varphi] d\, \beta_s^k$$

$$= M_t[\varphi].$$

Hence

$$M_t[\varphi] = \left(\int_0^t f_s \, dW_s\right)[\varphi]$$

for all $\varphi \in \Phi$, i.e.

$$M_t = \int_0^t f_s \, dW_s$$

$$= \int_0^t B_s(x_s) d\, W_s \ .$$

The existence of a weak solution in $C(\mathbb{R}_+;\Phi')$ follows immediately from (6.1) and (6.9):

Theorem 6.2. There exists a weak solution to the stochastic differential equation (1.1) on the canonical space $(\Omega, \underline{F}, P)$.

7. Existence and Uniqueness of a Strong Solution.

Definition 7.1. If for any two weak solutions (X^1, W) and (X^2, W) of (1.1) on the same interval $[0,T]$ and the same probability space $(\Omega, \underline{F}_t, P)$ with the same Wiener martingale W

$$P[\omega \in \Omega : X_t^1(\omega) = X_t^2(\omega),\ 0 \leq t \leq T] = 1,$$

we say that (1.1) has the *pathwise uniqueness property*.

The pathwise uniqueness property asserts that two weak solutions X^1 and X^2 on the same probability space, with respect to the same Wiener martingale, must be identical. The natural notion of uniqueness for weak solutions is not pathwise uniqueness but *distributional* uniqueness, i.e. uniqueness of the probability measure induced on the canonical path space by any weak solution; the following theorem, due to Yamada and Watanabe (See [4]), connects the two notions:

Theorem 7.1. (Yamada and Watanabe). Pathwise uniqueness implies distributional uniqueness for solutions to (1.1).

Proof. The idea of the proof is to induce probability measures P^i on the canonical space $\Omega := C(\mathbb{R}_+; \Phi' \times \Phi')$ (with the canonical filtration) giving the joint probability distribution of X^i and W^i, and to verify that each P^i can be factored as the product of Wiener measure $P(d\omega_2)$ on the second coordinate times a regular conditional probability distribution measure (RCPD) $P^i(d\omega_1|\omega_2)$ on the first coordinate. With these RCPDs it is possible to construct a measure $\tilde{P}(d\omega_1, d\omega_2, d\omega_3) := P^1(d\omega_1|\omega_3)\ P^2(d\omega_2|\omega_3)\ P(d\omega_3)$ on the space $\tilde{\Omega} := C(\mathbb{R}_+; \Phi' \times \Phi' \times \Phi')$ such that the two processes $X_i(\omega_1, \omega_2, \omega_3) := \omega_i$ $(i=1,2)$ are both solutions to (1.1) on the same probability space $(\tilde{\Omega}, \tilde{P})$ with respect to the same Wiener martingale $W(\omega_1, \omega_2, \omega_3) := \omega_3$. Pathwise uniqueness now implies that $\tilde{P}$ is

concentrated on the set $\{\omega_1, \omega_2, \omega_3 : \omega_1 = \omega_2\}$ and hence that the marginals P^i must be equal.

Although Yamada and Watanabe only state their result for R^d-valued processes, their proof remains valid for the Φ'-valued processes which concern is here; since Φ' is a standard space, the existence of regular conditional probability distributions presents no problem. ∎

Theorem 7.2. [4] Pathwise uniqueness and the existence of a weak solution together imply the existence of a unique strong solution.

Proof. See Theorem IV.1.1 of [4].

In view of the above result, it remains only to prove the pathwise uniqueness property for Equation (1.1). We are able to do this by adding the monotonicity condition (MC) to the conditions already assumed.

Theorem 7.3. Under the conditions (IC), (CC), (MC), (LG) and (JC) of Section 3, Equation (1.1) has the pathwise uniqueness property.

Proof. The argument here, closely follows [8] and we give it here only for the sake of completeness and the reader's convenience. By Theorems 5.2 and 6.1, Equation (1.1) has a weak solution in $C([0,T], H^{-m})$. Let $X^i_\cdot \in C([0,T], H^{-m})$ $(i=1,2)$ be two solutions. For convenience, set

$$Y_t := X^1_t - X^2_t, \; f_t := B_t(X^1_t) - B_t(X^2_t), \; I_t := \int_0^t f_s \, d W_s \quad \text{and} \quad \alpha_t := A_t.$$

Let $p \geq m$ be the integer mentioned in (MC). Recalling our notation that $(h^p_j) \subset \Phi$ is a CONS in H^p and applying Itô's formula to $(Y_t[h^p_j])^2$ we have

$$\sum_{j=1}^{\infty} (Y_t[h^p_j])^2 = 2 \int_0^t \sum_{j=1}^{\infty} Y_s[h^p_j]\alpha_s[h^p_j]ds + 2 \int_0^t \sum_{j=1}^{\infty} Q[f^*_s h^p_j, \; f^*_s h^p_j]ds$$

$$+ 2 \sum_{j=1}^{\infty} \int_0^t Y_s[h^p_j] \, dI_s[h^p_j]$$

which can be written as

$$|Y_t|^2_{-p} = 2 \int_0^t \{(Y_s, \alpha_s)_{-p} + |Q_{f_s}*|_{-p,-p}\}ds + M_t$$

where M_t is the continuous, local L^2-martingale represented by the last term in the above equation. An application of Itô's formula, this time to $|Y_t|^2_{-p}\, e^{-2\theta t}$ yields the relation

$$e^{-2\theta t}|Y_t|^2_{-p} = -2\theta \int_0^t |Y_s|^2_{-p}\, e^{-2\theta s} ds + 2 \int_0^t \{(Y_s, \alpha_{-s})_{-p} + |Q_{f_s}*|_{-p,-p}\}\, e^{-2\theta s}$$

ds

$$+ \int_0^t e^{-2\theta s}\, d\, M_s\ .$$

Using (MC) we have

$$e^{-2\theta t}\ |Y_t|^2_{-p} \le \eta_t$$

where $\eta_t := \int_0^t e^{-2\theta s}\, dM_s$ is a continuous, local martingale which is nonnegative (in view of the above inequality). Hence, for a sequence of stopping times $\sigma_n \uparrow \infty$, we have for every $\epsilon > 0$,

$$\epsilon\, P[\sup_{0 \le t \le T} \eta_{t \wedge \sigma_n} > \epsilon] \le E[\eta_{o \wedge \sigma_n}] = 0,$$

so that making $\sigma_n \uparrow \infty$ we obtain $\sup_{0 \le t \le T} \eta_t = 0$, P-a.s..

It follows that

$$|Y_t|^2_{-p} = 0 \quad \forall\ t \le T, \quad \text{P-a.s., i.e.}$$

$$\sup_{0 \le t \le T} |X^1_t - X^2_t|_{-p} = 0 \quad \text{P-a.s.,}$$

proving pathwise uniqueness for $C([0,T], H^{-p})$.

Note that Theorem 7.3 implies the pathwise uniqueness property for

solutions in $C([0,T],\Phi')$. For given two such solutions X^i, there exists a common index $m > 0$ such that $X^i \in C([0,T],H^{-m})$ for $i=1,2$. The proof of Theorem 7.3 then applies and the assertion is true.

The following result is an immediate consequence of Theorems 6.1, 7.1, 7.2, 7.3 and the above Remark. ∎

Corollary 7.3.1. For each $T > 0$ there exists a unique strong solution to Equation (1.1) on the interval $[0,T)$. By this we mean that for any probability space $(\Omega,\underline{F},P)$ on which are defined a Φ' random variable ξ with probability distribution measure $\mu_0 = P \circ \xi^{-1}$ and a Wiener martingale W with covariance quadratic form Q, if μ_0, Q, and the coefficient functions A and B all satisfy the conditions (IC), (CC), (MC), (LG) and (JC) of Section 3, then there exists a unique strong solution $X = \{X_t\}$ to Equation (1.1) for all $0 \leq t \leq T$.

Proof. Apply Theorems 6.1, 7.1, 7.2, and 7.3. ∎

We now come to our main result, the existence of a unique solution to (1.1) for all time:

Theorem 7.4. Let $(\Omega,\underline{F},P)$ be a probability space on which are defined a Φ' random variable ξ with probability distribution measure $\mu_0 = P \circ \xi^{-1}$ and a Wiener martingale W with covariance quadratic form Q, and suppose that μ_0, Q, and the coefficient functions A and B all satisfy the conditions (IC), (CC), (MC), (LG) and (JC) of Section 3. Then there exists a unique strong solution X $= \{X_t\}$ to Equation (1.1) for all $0 \leq t < \infty$.

Proof. Use Corollary 7.3.1 to construct a strong solution X_t^T with starting value ξ and Wiener martingale W on the interval $[0,T)$ for $T = 1,2,3,\ldots$; verify that the definition

$$X_t := \sum_{n=1}^{\infty} 1_{[n-1,n)}(t) X_t^n$$

determines a strong solution for all time. Uniqueness follows from that of each X_t^n. ∎

Remark. Suppose that H is a separable Hilbert space with inner product $(\ ,\ )_0$ on which is defined a continuous, contraction semigroup with generator $-A$ satisfying the following properties:

(i) For some $r_1 > 0$, $(I + A)^{-r_1}$ is a Hilbert-Schmidt operator on H. Then A has a discrete spectrum with eigenvalues and eigenfunctions given by $A\varphi_j = \lambda_j \varphi_j$ ($\lambda_j > 0$) with

(ii) $\sum_j (1 + \lambda_j)^{-2r_1} < \infty$.

This set up occurs in many physical and biological problems (see [6] and the references given therein). A convenient Fréchet nuclear space Φ can then be defined:

$$\Phi = \{\varphi \in H : \sum_{j=1}^{\infty} (1 + \lambda_j)^{2r} (\varphi, \varphi_j)_0^2 < \infty \qquad \forall\ r \geq 0\}.$$

For $\varphi \in \Phi$, define the Hilbertian norms

$$|\varphi|_r^2 = \sum_{j=1}^{\infty} (1 + \lambda_j)^{2r} (\varphi, \varphi_j)_0^2$$

Let $H^r = |\cdot|_r$-completion of Φ. Then Φ is a countably Hilbertian nuclear space as can easily be verified. Further, H^{-r} is the dual of H^r, $\Phi = \bigcap_{r>0} H^r$ and $\Phi' = \bigcup H^{-r}$. The stochastic differential equation (1.1) becomes *quasilinear* with $A_t(u) = -A^* u$ ($u \in \Phi'$) where A^* is the adjoint of A restricted to Φ. (It can be checked that $A\Phi \subseteq \Phi$) and linear if, in addition, $B_t(u) \equiv I$. For any $m \geq$

1 and $u \in H^{-m}$ it is easy to verify that $A^{*}u \in H^{-p}$ for $p \geq m+1$. Further since $|A\varphi|_r \leq |\varphi|_{r+1}$ for all $r > 0$, if $u \in H^{-m}$ we have

$$|A_t(u)[\varphi]| = |-A^{*}u[\varphi]| = |u[-A\varphi]| \leq |u|_{-m}\, |\varphi|_{m+1}$$

and hence

$$|A_t(u)|_{-p} \leq |u|_{-m} \quad \text{for} \quad p \geq m+1.$$

Thus the first part of (LG) is satisfied.

Also, the drift coefficient "helps" the monotonicity condition because for $u \in H^{-m}$ and $p \geq m+1$,

$$(A_t(u),u)_{-p} = -(A^{*}u,u)_{-p} = -\sum_{j=1}^{\infty} \lambda_j (u,h_j^{-p})_p^2 \leq 0$$

where $h_j^p := (1+\lambda_j)^{-p}\varphi_j$ is a CONS in H^p. It is further to be noted that for this class of nuclear spaces Φ, our basic condition (A) is satisfied since (φ_j) is a common orthogonal system for all the spaces H^r.

8. Application to random strings. Funaki [3] has studied the random motion of strings by appealing to the theorem on the existence of a unique solution of the following nonlinear stochastic evolution equation on a separable Hilbert space H.

$$(8.1) \qquad \begin{cases} d\,X_t = a(t,X_t)dB_t + b(t,X_t)dt - AX_t dt\ , \quad t \in [0,T], \\ X_0 \in H, \end{cases}$$

where B_t is a cylindrical Brownian motion, the coefficients $a(t,x)$ and $b(t,x)$ satisfy suitable Lipschitz continuity conditions and A is a non-negative, self-adjoint operator on H with pure point spectrum $0 \leqq \lambda_1 \leqq \lambda_2 \leqq \ldots$ such that $\lambda_k \sim c\, k^{1+\delta}$, $(c,\ \delta > 0)$ as $k \to \infty$.

In this section, we will show how Theorem 7.4 can be used as an alternative approach in a similar setting. Let A be a non-negative self-adjoint operator on H with pure point spectrum $0 \leqq \lambda_1 \leqq \lambda_2 \leqq \ldots$ such that $\sum_{j=1}^{\infty} (1 + \lambda_j)^{-2r} < \infty$ for some $r > 0$. Let us now work with the Φ introduced in the Remark at the end of Section 7.

We will assume conditions on the coefficients of the stochastic evolution equation

$$(8.2) \qquad d\,X_t = A_t(X_t)dt + B(t,X_t)d\,W_t,$$

namely, there exist a sufficiently large m, numbers $\theta > 0$ and $p > m$ such that $A_t(x)$ and $B_t(x)$ satisfy the conditions (CC), (MC) and (LG).

Then by appealing to Theorem 5.3 and Theorem 7.4, for the support of the solution X_t of (8.2), we get

$$P\Big[\bigcup_{n=1}^{\infty} \{ \sup_{0 \leq t \leq n} |X_t|_{-m} < \infty\}\Big] = 1.$$

Hence from the weak continuity of X_t, $\sum_{j=1}^{\infty} (1+\lambda_j)^{-2r} < \infty$ and the Lebesgue convergence theorem, we obtain

Corollary 8.1. Under the conditions on the coefficients $A_t(x)$ and $B_t(x)$ and the initial value condition (IC) of Theorems 5.2 and 7.4, (8.2) has a unique solution X_t such that $X_{\bullet} \in C(\mathbb{R}_+, H^{-(m+r)})$.

Now, we will consider the stochastic evolution equation of Funaki's type. If $\varphi \in \Phi$, then we have for all $\ell \geq 0$

$$\sum_{j=1}^{\infty} (1 + \lambda_j)^{2\ell} (A\varphi, \varphi_j)_0^2$$

$$= \sum_{j=1}^{\infty} (1 + \lambda_j)^{2\ell} \lambda_j^2 (\varphi, \varphi_j)_0^2$$

$$\leqq |\varphi|_{\ell+1}^2 < \infty,$$

so that $A\varphi \in \Phi$. Let A^* be the adjoint of A with respect to the canonical bilinear form $\langle , \rangle$ on $\Phi' \times \Phi$. Let

(8.3) $$d X_t = a(t,X_t) d W_t + b(t,X_t)dt - A^* X_t \, dt,$$

where W_t is a Φ'-valued Wiener process such that $E[W_t[\varphi]^2] = t\, Q[\varphi,\varphi]$ and $Q[\varphi,\varphi]$ is a positive definite continuous quadratic form on Φ.

Before proceeding to the assumptions on the coefficients $a(t,x)$ and $b(t,x)$, we notice that there exist an integer q and a constant C_1 such that

(8.4) $$|Q[\varphi,\varphi]| \leqq C_1 |\varphi|_q^2 .$$

From now on, we denote positive constants by C_i, $i=2,3,\ldots$. Since $\sum_{j=1}^{\infty} (1+\lambda_j)^{-2r} < \infty$, the support of W_t is contained in $H^{-(q+r)}$, [7].

Let $a(t,x) : R_+ \times \Phi' \to L(\Phi',\Phi')$ and $b(t,x) : R_+ \times \Phi' \to \Phi'$ be jointly

continuous mappings of (t,x) satisfying the following conditions:

There exist $q' \geqq q+r$ and $m \geqq q'$ such that $a(t,x)$ maps $H^{-(q+r)}$ to $H^{-q'}$, $b(t,\cdot)$ maps H^{-m} to H^{-m} and for $x,y \in H^{-m}$,

$$\text{(A.1)} \qquad \sum_{j=1}^{\infty} |a(t,x)h_j^{-q}|_{-q'}^2 \leqq K(1 + |x|_{-m}^2),$$

$$\sum_{j=1}^{\infty} |a(t,x)-a(t,y))h_j^{-q}|_{-q'}^2 \leqq K|x-y|_{-m}^2 .$$

and

$$\text{(A.2)} \qquad |b(t,x)|_{-m}^2 \leqq K(1+|x|_{-m}^2),$$

$$|b(t,x) - b(t,y)|_{-m} \leqq K|x-y|_{-m} .$$

where $h_j^m = \varphi_j/(1+\lambda_j)^m$ and K is a constant.

To apply Theorem 7.2, it is enough to check the conditions (CC), (MC) and (LG).

Setting $A_t(x) = -A^*x + b(t,x)$ and $B_t(x) = a(t,x)$, for $x \in H^{-m}$, we have

$$\begin{aligned} |A^*x|_{-(m+1)}^2 &= \sum_{j=1}^{\infty} A^*x[h_j^{m+1}]^2 \\ &= \sum_{j=1}^{\infty} x[Ah_j^{m+1}]^2 = \sum_{j=1}^{\infty} x[\lambda_j\varphi_j/(1+\lambda_j)^{m+1}]^2 \\ &\leqq \sum_{j=1}^{\infty} x[\varphi_j/(1+\lambda_j)^m]^2 = \sum_{j=1}^{\infty} x[h_j^m]^2 = |x|_{-m}^2 \end{aligned}$$

and hence, together with condition (A.2), we get

$$\text{(8.5)} \qquad |A_t(x)|_{-(m+1)}^2 \leqq C_2(1+|x|_{-m}^2).$$

By (8.4) and (A.1), we have

$$\text{(8.6)} \qquad |Q_{B_t(x)}|_{-m,-m} = \sum_{j=1}^{\infty} Q(a(t,x)^* h_j^m, a(t,x)^* h_j^m)$$

$$\leqq C_1 \sum_{j=1}^{\infty} |a(t,x)^* h_j^m|_q^2$$

$$= C_1 \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} h_k^{-q} [a(t,x)^* h_j^m]^2$$

$$= C_1 \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a(t,x) h_k^{-q} [h_j^m]^2$$

$$= C_1 \sum_{k=1}^{\infty} |a(t,x) h_k^{-q}|_{-m}^2$$

$$\leq C_1 \sum_{k=1}^{\infty} |a(t,x) h_k^{-q}|_{-q}^2.$$

$$\leqq C_3 (1 + |x|_{-m}^2).$$

Hence (8.5) and (8.6) yield the condition (LG).

Suppose $x, y \in H^{-m}$. Since $-A^* x \in H^{-(m+r)}$, we get

$$(8.7) \qquad (-A^* x, x)_{-(m+r)} = \sum_{j=1}^{\infty} (x, h_j^{-(m+r)})_{-(m+r)} (A^* x, h_j^{-(m+r)})_{-(m+r)}$$

$$= \sum_{j=1}^{\infty} x[h_j^{m+r}]\, x[A\, h_j^{m+r}]$$

$$= -\sum_{j=1}^{\infty} \lambda_j\, x[h_j^{m+r}]^2 \leqq 0.$$

By (8.7) and (A.2), we have

$$(8.8) \qquad 2(A_t(x) - A_t(y), x-y)_{-(m+r)}$$

$$= 2(-A^*(x-y), x-y)_{-(m+r)} + 2(b(t,x) - b(t,y), x-y)_{-(m+r)}$$

$$\leqq 2(b(t,x) - b(t,y), x-y)_{-(m+r)}$$

$$= 2 \sum_{j=1}^{\infty} (x-y) [h_j^m](b(t,x) - b(t,y),\ h_j^{-m})_{-(m+r)}$$

$$= 2 \sum_{j=1}^{\infty} (x-y)[h_j^m] \sum_{k=1}^{\infty} (b(t,x) - b(x,y))\ [h_k^m]\ (h_k^{-m}, h_j^{-m})_{-(m+r)}$$

$$= 2 \sum_{j=1}^{\infty} (x-y)[h_j^m]\ (b(t,x) - b(x,y))\ [h_j^m]\ |h_j^{-m}|^2_{-(m+r)}$$

$$\leqq C_4 |x-y|_{-m}\ |b(t,x) - b(t,y)|_{-m}$$

$$\leqq C_4\ K\ |x-y|^2_{-m}\ .$$

Using (A.1) and an estimation procedure similar to that of (8.6), we get

(8.9) $$|Q_{B_t(x)-B_t(y)}|_{-m,-m} \leqq C_5 |x-y|^2_{-m},\ 0 \leqq t \leqq T\ .$$

Therefore (8.8) and (8.9) imply the condition (MC).

The inequalities (A.2) (8.6) and (8.7) yield the condition (CC), which completes the assertion. Therefore by Corollary 8.1, the equation (8.3) has a unique solution X_t such that $X_\bullet \in C(R_+, H^{-(m+r)})$.

Now we specialize to Funaki's spectral condition: $\lambda_k \sim c\ k^{1+\delta}$, $(c,\ \delta > 0)$. We replace the cylindrical Brownian motion (with $Q[\varphi\ ,\ \varphi] = |\varphi|^2_0$), by a Wiener process W in $C(\mathbb{R}_+,\ H^{-r})$, $r > 1/2(1+\delta)$. Using Corollary 8.1 we conclude that (8.3) has a unique solution X_t such that

$$x \in C(\mathbb{R}_+,\ H^{-s}),$$

where $s > m + r$, and $m + r > 2r > \frac{1}{(1+\delta)}$.

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